Proceedings of the Harriett J. Walton Symposium on Undergraduate Mathematics Research

Volume 11

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Preface

The Department of Mathematics initiated the Harriett J. Walton Symposium on Undergraduate Mathematics Research to encourage undergraduate students in mathematics research and practice. We believe that undergraduate research experiences must count among the most challenging and rewarding experiences for college students. To research and present material beyond what is traditionally covered in the classroom to explore mathematics independently may be considered the best career preparation for students regardless of their post-college plans. This yearly symposium on undergraduate research is dedicated to Professor Harriett J. Walton who served forty-two years on the faculty of Morehouse College.

The Eleventh Annual Harriett J. Walton Symposium on Undergraduate Mathematics Research was held on the campus of Morehouse College in Atlanta, Georgia, USA on April 6, 2013. Twelve students from Albany State University, eleven students from Birmingham Southern College, eight students from Morehouse College, five students from Clayton State University, three students from Georgia State University, two students from Paine College, and one student from Spelman College gave presentations on their research and studies in mathematics and related fields. The Symposium was sponsored by the Department of Mathematics and the Division of Science and Mathematics of Morehouse College through the generous support of The Mathematical Association of America (MAA) Regional Undergraduate Mathematics Conference Program through National Science Foundation Grant DMS-0846477. This volume contains six articles and twenty abstracts submitted by the Symposium participants and their advisors.

The organizers of the Symposium thank the presenters and their advisors for preparing a remarkable collection of lectures for the Symposium. We thank the referees for their service to evaluate and improve the papers before their publication. We thank the administration of Morehouse College for their generous support, especially Dr. J. K. Haynes, Dean of the Division of Science and Mathematics, Dr. Garikai Campbell, Provost and Senior Vice President for Academic Affairs, and President Dr. John Wilson Jr. We thank the MAA for their support, advice, and materials for the Symposium. Special thanks to Morehouse College Students and Mr. William Barnville for coordinating many aspects of the Symposium. Finally we thank Professor Walton for attending the Symposium.

Professor Harriett J. Walton

In September 1958, Harriett J. Walton joined the faculty of Morehouse College during the presidency of Benjamin Elijah Mays. She became a member of a team of three persons in the Department of Mathematics where she worked with the legendary Claude B. Dansby who served as Department Chair. Dr. Walton and her two colleagues taught all of the mathematics for the majors as well as the mathematics for non-science students. Dr. Walton relates that two of her favorite courses that she taught during this period were Abstract Algebra and Number Theory. The three member mathematics department did an excellent job of preparing their mathematics majors for graduate school and the other students for success in their respective disciplines. In fact it was during this period of history that Morehouse gained the reputation of being an outstanding Institution especially for African American men. As the department grew, Dr. Walton shifted her attention away from mathematics majors and began to concentrate on students who needed special attention and care in order to succeed in mathematics. She became an advisor, mentor, tutor and nurturer to a large number of students matriculating at Morehouse College. Because of the caring attitude that she had for her students, some of them to this day refer to her as "Mother Walton".

Dr. Walton has never been satisfied with mediocrity. Throughout her teaching career she demonstrated a love for learning. In 1958 when she arrived at Morehouse College she had an undergraduate degree in mathematics from Clark College in Atlanta, Georgia, a Master of Science degree in mathematics from Howard University, Washington D.C., and a second Master's degree in mathematics from Syracuse University. While at Morehouse teaching full time and raising a family of four children, Dr. Walton earned the Ph.D. degree in Mathematics Education from Georgia State University. After receiving her doctorate, Dr. Walton realized the emerging importance of the computer in education so she returned to school and in 1989 earned a Master's degree in Computer Science from Atlanta University. She is indeed a remarkable person.

Dr. Walton's list of professional activities, awards and accomplishments during her career is very impressive and too lengthy to be enumerated here. However a few special ones are her memberships in Alpha Kappa Mu, Beta Kappa Chi, Pi Mu Epsilon, and the prestigious Phi Beta Kappa Honor Society. Additionally she was selected as a Fulbright Fellow to visit Ghana and Cameroon in West Africa. Dr. Walton's professional memberships included the American Mathematical Society, the Mathematical Association of America, National Council of Teachers of Mathematics (NCTM) and the National Association of Mathematicians (NAM). She served as Secretary/Treasurer of NAM for ten years. In May 2000, Dr. Walton retired from Morehouse College after forty-two (42) years of service.

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Integral Equations of Fredholm Type

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Abstract In this study, we investigate solutions of linear integral equations of Fredholm type with finite rank kernels. We then use kernels of finite rank to approximate solutions of linear integral equations of Fredholm Type with continuous kernels and give a relative error bound on the approximation. This method can be applied to approximate solutions of a larger class of linear integral equations of Fredholm type, namely those with L^2 kernels.

1. Introduction

Many physical science phenomena can be modeled by differential equations. For example, harmonic motion, and how to relate an unknown function to its derivative. Other applications are in Exponential growth and decay, falling objects Newtons law of cooling, and RL circuits. Because many times differential equations can be tedious to solve we transform them into linear equations.

We will discuss initial value problems specifically . In this type of problem we are given a differential equation and an initial value.

$$\begin{cases} y'(x) &= f(x,y) \\ y(0) &= y_0 \end{cases}$$

We then must convert the differential equation to an integral equation by first integrating both sides with respect to the dummy variable s, $\int_0^t \frac{dx(s)}{ds} ds = \int_0^t f(s,x) ds$. We obtain

$x(t) - x(0) = \int_0^t f(s, x(s)) ds$. Hence,

$x(t) = x_0 + \int_0^t f(s, x(s)) ds$.

This type of equation is known as an integral equation of Volterra type. Integral equations of Fredholm type with a kernel $K(s,t)$ are $x(s) = x(0) + \int_a^b K(s,t)x(t)dt$.

We choose to convert differential equations to integral equations because they are more practical to answer existence questions and to approximate solutions.

2. Types Of Integral Equations

An integral equation is an equation in which an unknown function appears under the integral sign. Here are some examples of integral equations:

$L(x(s)) = y(s) = \int_a^b K(s,t)x(t)dt$, $(a \leq s \leq b)$. (2.1)

$L(x(s)) = x(s) = y(s) + \int_a^b K(s,t)x(t)dt$, $(a \leq s \leq b)$ (2.2)

$L(x(s)) = x(s) = \int_a^b K(s,t)[x(t)]^2 dt$, $(a \leq s \leq b)$ (2.3)

$L(x(s)) = x(s) = \int_a^b \int_a^b K(s,t,u)x(t)x(u)dtdu$, $(a \leq s \leq b)$. (2.4)

In this project, we will study solutions of integral equations like those in (2.2).

2.1. **Linear Integral Equations.**

Definition 2.1. *An integral equation is said to be linear if the unknown function appears to the power one inside the integral sign.*

For example, equations (2.1) and (2.2)above are linear because they contain an unknown in the integral raised to the first power and are of the form of a linear equation. But (2.3) and (2.4) are not linear because (2.3) has the unknown raised to the second power and (2.4) has multiple unknowns under the radical.

A linear equation of the first kind has a variable that appears only inside the integral. But a linear equation of the second kind contains a variable that appears on the inside of the integral and on the outside. We dealt primarily with linear equations of the second kind.

$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$ is a linear equation of Fredholm type with λ serving as the parameter associated with this integral equation.

The equation $x(s) = y(s) + \lambda \int_a^s K(s,t)x(t)dt$ where $(a \leq s \leq b)$ is an integral equation of Volterra type.

2.2. **Converting Differential Equations to Integral Equations.** The first step in solving a difficult differential equation is to change it from a differential equation to an integral equation. Consider the second order differential equation with the given initial conditions. This could model the motion of an object subject to initial position and initial velocity.

$$\begin{cases} y''(x) &= f(x,y) \\ y(0) &= y_0 \\ y'(0) &= y_1 \end{cases}$$

First we take the integration limits from 0 to x $\int_0^x \frac{d^2y(u)}{du^2} = \int_0^x f[u, y(u)]du$ then we solve the integral to obtain the resulting equation $y'(x) = y_1 + \int_0^x f[u, y(x)]du$.

We now have a first order integral $y'(x) = y_1 + \int_0^t f[u, y(x)]du$. To obtain a linear equation of the second kind we must integrate both sides once again

$\int_0^x y'(t)dt = \int_0^x y_1 dt + \int_0^x (\int_0^t f[u, y(u)]du)dt$, we then evaluate the integrals to obtain $y(x) - y(0) = y_1(x-0) + \int_0^x (\int_0^t f[u, y(u)]du)dt$. We had to change the integrations limits to be from u to x inside the parenthesis and solve these integrals.

$y(x) = y_0 + y_1(x) + \int_0^x (\int_u^x f[u, y(u)]dt)du$

$y(x) = y_0 + y_1(x) + (\int_0^x f[u, y(u)] \int_u^x dt)du$

$y(x) = y_0 + y_1(x) + \int_0^x (x-u)f[u, y(u)]du$

This is a linear integral equation of the second kind also called the Volterra type.

Another model could be a vibrating string attached at both ends which yields a linear integral equation of Fredholm type.

3. Existence of a Solution, Regular Value, and Resolvent Kernel

We now are working specifically with linear integral equations of Fredholm type of the second kind that follow previously mentioned form $x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$ where, λ is a parameter which may be a complex number, $K(s,t)$ is an L^2 kernel and $y(s)$ is an L^2 function.

A function x is L^2 if $\int_a^b |x(t)|^2 dt < \infty$ and $K(s,t)$ is L^2 if $\int_a^b \int_a^b |K(s,t)|^2 ds dt < \infty$.

Notations: $||x|| = ||x||_2 = \left(\int_a^b |x(t)|^2 dt\right)^{1/2}$ and $||K|| = ||K||_2 = \left(\int_a^b \int_a^b |K(s,y)|^2 ds dt\right)^{1/2}$.

Before proceeding, we use the operator equation $x = y + \lambda Kx$ to represent the above integral equation.

Continuing on with this method of thinking we write the solution (if it exists) as $x = y + \lambda Hy = (I + \lambda H)y$ where H is an operation that depends on λ and K.

So by substituting for $x = (I + \lambda H)y$ we obtain $(I + \lambda H)y = y + \lambda K(I + \lambda H)y = y + \lambda KIy + \lambda^2 KHy = \lambda Ky + \lambda^2 KHy + y = y + \lambda Hy$. This gives the operator equation $\lambda H = \lambda K + \lambda^2 KH$, which if $\lambda \neq 0$ gives

$$H - K = \lambda KH \tag{3.1}$$

Moreover if $x = (I + \lambda H)y$ is a unique solution of the integral equation then we substitute in for x and we solve the following equation $x = (I + \lambda H)y = (I + \lambda H)(I - \lambda K)x$
$Ix = (I - \lambda K + \lambda H - \lambda^2 HK)x$

$I = I - \lambda K + \lambda H - \lambda^2 HK$
So,$H - K = \lambda KH = \lambda HK$. Hence $\lambda HK = \lambda KH$

Definition 3.1. *If for a given value of the parameter λ there is an L^2 kernel $H(s,t)$ which satisfies $H - K = \lambda KH = \lambda HK$. Then $H(s,t)$ is called a resolvent kernel of the kernel $K(s,t)$ for the value λ and λ is called a regular value of the kernel $K(s,t)$. The equation $H - K = \lambda KH = \lambda KH = \lambda HK$ is called the resolvent.*

We note that $\lambda = 0$ is always a regular value since $x(s) = y(s)$. In that case let us consider the equation $x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$, we then write this equation as $x = y + KX$ for $(a \leq s \leq b)$.

We then write the equation in the form $x = (I + \lambda H)y$ if for some operator H. Then H satisfies $H - K = \lambda KH = \lambda HK$
$x(s) = y(s) = \lambda \int_a^b H(s,t)y(t)dt$

Theorem 3.2. *If, for a given value of λ, a resolvent kernel of the L^2 kernel K exists, then it is unique.*

Proof: We must first suppose that
$H_1 - K = \lambda H_1 K = \lambda K H_2$ $H_2 - K = \lambda H_2 K = \lambda K H_2$

We can then write it as $H = H_1 - H_2$. By subtracting the two equations we obtain the form $H = \lambda KH$

Now we obtain the equation $H_1 H = H_1(\lambda HK) = (\lambda H_1 K)H = (H_1 - K)H = H_1 H - KH$ so that $KH = 0$, and therefore, by $H = \lambda KH$, $H = 0$, i.e. $H_1 = H_2$.

Theorem 3.3. *Let λ be a regular value of the L^2 kernel K, and let H be the corresponding resolvent kernel. If y is a given L^2 function, the equation $x = y + \lambda Kx$ has a unique L^2 solution x given by $x = y + \lambda Hy$.*

From this theorem we find that x is defined by the aforementioned equation $x = y + \lambda Hy$. From this knowledge we can substitute in $y + \lambda Hy$ (which is the inverse of x) for x and solve the equation

$$y + \lambda K(y + \lambda Hy) = y + \lambda Ky + \lambda^2 KHy = y + \lambda Ky + \lambda(H - K)y = y + \lambda Hy = x$$

because x is a solution of the previous equation. This also is true if x satisfies the previous equation we have $y = x - \lambda Kx$. Now we know that

$$y + \lambda HY = x - \lambda Kx + \lambda Hx\lambda^2 HKx = x + \lambda(H - K)x - \lambda(H - K)x = x,$$

thus x should be given by the equation $x = y + \lambda Hy$ and the solution is unique.

3.1. **Solutions of Linear Integral Equations of Fredholm Type of second order.** Although we cannot find the exact solution to linear integral equations of the second order because the unknown is found within the signs of integration; we can approximate the solutions for equations of the form $x(s) = y(s) + \lambda K(s,t)x(t)dt$ for all $(a \leq s \leq b)$ where $y(s)$ is an L^2 function and $K(s,t)$ is an L^2 kernel. We start with a special case of L^2 kernels $K(s,t)$, namely $K(s,t) = \sum_{i=1}^{n} a_\nu(s)b_\nu(t)$ for (a $\leq$ s,t $\leq$ b) where a_ν, $b_\nu(t)$ are L^2 functions for $\nu = 1, 2, ..., n$. These types of kernels are called kernels of finite rank. We denote them by $K = \sum_{i=1}^{n} a_\nu \otimes b_\nu$ and if $a_1, a_2...a_n$ as well as $b_1b_2...b_n$ are linearly independent, then we say that K has finite rank n.

Notation:
$x_\mu = (x, b_\mu) = \int_a^b x(t)b_\mu(t)dt$

$y = b_\mu = \int_a^b y(t)b_\mu(t)dt$

$(a_\mu, b_\mu) = K\mu = \int_a^b a_\nu(t)b_\nu dt.$

3.2. **Example.** Now we will attempt to solve a linear equation of Fredholm type. We begin with an equation of the form $x(s) = s^2 + 2\int_0^1 (st + s^2t^2)x(t)dt$

In order to solve that equation we must first find $x(t)$. In order to find $x(t)$ we must first formulate the following equations and solve them to form a system of linear equations.
$y_1 = (y, b_1) = \int_0^1 t(t^2)dt = \frac{1}{4}$
$y_2 = (y, b_2) = \int_0^1 t^2t^2dt = \frac{1}{5}$
$k_{11} = (a_1, b_1) = \int_0^1 t(t)dt = \frac{1}{3}$
$k_{12} = (a_2, b_1) = \int_0^1 t^2(t)dt = \frac{1}{4}$
$k_{21} = (a_1, b_2 = \int_0^1 t(t^2)dt = \frac{1}{4}$
$k_{22} = (a_2, b_2) = \int_0^1 t^2t^2dt = \frac{1}{5}$

now that we have solved the integral equations we can now substitute them into the following system of linear equations

$$\begin{pmatrix} x_1 &= y_1 + 2k_{11}x_1 + 2k_{12}x_2 \\ x_2 &= y_2 + 2k21x_1 + 2k_{22}x_2 \end{pmatrix}$$

After solving the system of equations and substituting back into the original linear equations we find that $x_1 = -5$ and $x_2 = \frac{-23}{6}$.
We then substitute the values for x_1 and x_2 into the equation $x(s) = s^2 + 2x_1a_1(s) + 2x_2a_2(s)$ to find the value of $x(s)$. Next formulated a new equation for $x(s)$ $x(s) = -\frac{20}{3}s^2 - 10s$. Now we insert the equation for $x(s)$ into the equation of $x(t)$ and check to see if the solution is correct which, in our case, the solution is correct.

3.3. **Solutions of integral equations with kernels of finite rank.**

Theorem 3.4. *Let $K = \sum_{\nu=1}^{n} a_\nu \otimes b_\nu$ be an L^2 kernel of finite rank, let y be an L^2 function and write $(y, b_\mu) = y_\mu$, $(a_\nu, b_\mu) = k_\mu nu$, $(1 \leq \mu \leq n,\ 1 \leq \nu \leq n)$. If x is an L^2 solution of the integral equation $x = y + \lambda Kx$,and $x_\mu = (x, b_\mu)$, $(\ 1 \leq \mu \leq n)$, then $x_\mu = y_\mu + \lambda \sum_{\nu=1}^{n} k_\mu \nu x_\nu$, $(1 \leq \mu \leq n)$. Conversely, if (x_μ) is a solution of the system of*

linear equations and $x = y + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu(s)$ *then* x *is an* L^2 *solution of* $x = y + \lambda K x$ *and* $(x, b\mu) = x_\mu$ *for* $mu = 1, ...n$.

Proof: Assume that $x(t)$ is a solution of the $x = y + \lambda K x$ and $x_\mu = (x, b_\mu)$ then

$$x(s) = y(s) + \lambda \int_a^b \sum_{\nu=1}^{n} a_\nu(s) b_\nu(t) x(t) dt.$$

So

$$x(s) = y(s) + \lambda \sum_{\nu=1}^{n} \int_a^b a_\nu(s) b_\nu(t) x(t) dt = y(s) + \lambda \sum_{\nu=1}^{n} a_\nu(s) \int_a^b x(t) b_\nu(t) dt =$$

$$y(s) + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu(s)$$

.

Next, $x = y + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu$, therefore $(x, b_\mu) = (y + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu, b_\mu) = (y, b_\mu) + \lambda \sum_{\nu=1}^{n} x_\nu (a_\nu, b_\mu)$

and thus

$$x_\mu = (x, b_\mu) = \int_a^b y(t) b_\mu(t) dt + \lambda \sum_{\nu=1}^{n} x_\nu \int_a^b a_\nu(t) b_\mu(t) dt = y_\mu + \lambda \sum_{\nu=1}^{n} x_\nu \int_a^b a_\nu(t) b_\mu(t) dt.$$

Therefore $x_\mu = y_\mu + \sum_{\nu=1}^{n} k_{\mu\nu} x_\nu$, $1 \leq \mu \leq n$.

Conversely, assume that $(x_\mu) = x_1, x_2 .... x_n$ is a solution of $x_\mu = y_\mu + \lambda \sum_{\nu=1}^{n} k_{\mu\nu} x_\nu$, where $1 \leq \mu \leq n$ and let x be defined by $x(s) = y(s) + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu(s)$. Then $(x, b_\mu) = \int_a^b x(t) b_\mu(t) dt$ $= \int_a^b \left(y(t) + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu(t)\right) b_\mu(t) dt = \int_a^b y(t) b_\mu(t) dt + \lambda \sum_{\nu=1}^{n} x_\nu \int_a^b a_\nu(t) b_\mu(t) dt = y_\mu + \lambda \sum_{\nu=1}^{n} x_\nu k_{\mu\nu}$ $= x_\mu$. Thus $(x, b_\mu) = x_\mu$

So $x(s) = y(s) + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu(s) = y(s) + \lambda \sum_{\nu=1}^{n} (x, b_\nu) a_\nu(s) = y(s) + \lambda \sum_{\nu=1}^{n} \int_a^b a_\nu(s) b_\nu(t) x(t) dt =$ $y(s) + \lambda \int_a^b \sum_{\nu=1}^{n} a_\nu(s) b_\nu(t) x(t) dt = y(s) + \int_a^b K(s,t) x(t) dt$ which solves $x = y + \lambda K x$.

4. Approximations with finite kernels

Before stating the main result we state two useful facts, namely the Wieirstrass Approximation Theorem and Fubini's Theorem.

The Wieirstrass Approximation Theorem states let F be continuous on $[a, b] \times [a, b]$ with values in $\mathbb{R}$. Then F can be approximated by polynomials and $[a, b] x [a, b]$ to any degree of tolerance.$|F(x, y) - P(x, y)| < \epsilon$ for all (x, y) in $[a, b] x [a, b]$.

Next, Fubini's Theorem states that if $\int \int f(s,t) ds dt$ exists as a Lebesgue integral and $\int f(s,t)$ dt exists and is an integrable. An example of this theorem is this equation $\int \int f(s,t) ds dt = \int ds \int f(s,t) dt$.

Theorem 4.1. *Let* $k(s,t)$ *be continuous* $[a, b] x [a, b]$ *and let* $\epsilon > 0$ *be arbitrary. Then there is a kernel* $k_f(s,t)$ *of finite rank such that* $||k - k_f|| < \epsilon$.

Proof: By the Weirstrass approximation theorem
There is a polynomial function $K(s,t)$ of degree such that $||k - k_f|| < \frac{\epsilon}{\lambda}$. Since k_f is a polynomial in s and t, it is of finite rank.

Theorem 4.2. *In equation* $x(s) = y(s) + \lambda \int_a^b K(s,t) x(t) dt$ *let* $K(s,t)$ *be continuous as* $[a, b] \times [a, b]$ *and let* $\epsilon > 0$ *and* $K_f(s,t)$ *be a kernel of finite rank* $||K - K_f|| < \frac{\epsilon}{\lambda}$. *Now let* $x(s)$ *be a solution corresponding to* K_f, *then* $\frac{||x - x_f||}{||x||} < \epsilon$. *That is to say that the relative error in approximating* $x(s)$ *by* $x_f(s)$ *can be arbitrarily small.*

Proof: Let $x(s)$ be a solution of $x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt$ and $x_f(s)$ be a solution of $x(s) = y(s) + \lambda \int_a^b K_f(s,t)x(t)dt$. Then for $a \leq s \leq b$ and by subtracting we obtain $x(s) - x_f = y(s) + \lambda \int_a^b K(s,t)x(t)dt - y(s) - \lambda \int_a^b K_f(s,t)x(t)dt$.
So $x(s) - x_f(s) = \lambda \int_a^b K(s,t)x(t)dt - \lambda \int_a^b K_f(s,t)x(t)dt$. Finally we obtain the equation $x(s) - x_f(s) = \lambda \int_a^b (K(s,t) - K_f(s,t)x(t)dt$. Then taking the absolute value of each side then inside the integral on the right yields the following

$$|x(s) - x_f(s)| = |\lambda| | \int_a^b (K(s,t) - K_f(s,t))x(t)dt| \leq |\lambda| \int_a^b |K(s,t) - K_f(s,t)||x(t)|dt.$$

By the Cauchy-Schwarz inequality we obtain

$$|x(s) - x_f(s)| \leq |\lambda| \left(\int_a^b |K(s,t) - K_f(s,t)|^2 dt \right)^{1/2} \left(\int_a^b |x(t)|^2 dt \right)^{1/2},$$

so $|x(s) - x_f(s)|^2 \leq |\lambda|^2 (\int_a^b |K(s,t)K_f(s,t)|^2 dt)(\int_a^b |x(t)|^2 dt)$
By integrating with respect to s on $[a,b]$ we obtain the inequality

$$\int_a^b |x(s) - x_f(s)|^2 ds \leq |\lambda| \int_a^b (\int_a^b |K(s,t) - K_f(s,t)|dt)(\int_a^b |x(t)|^2 dt)ds$$

By Fubinis Theorem we can pass from two iterated integrals to a single double integral

$$\int_a^b |x(s) - x_f(s)|^2 \leq |\lambda|^2 \left(\int_a^b \int_a^b |K(s,t) - K_f(s,t)|^2 dtds \right) \left(\int_a^b |x(t)|^2 dt \right).$$

Hence $||x - x_f||^2 \leq |\lambda|^2 ||K - K_f||^2 ||x||^2$.
Since $||K - K_f||^2 < \frac{\epsilon}{|\lambda|}$, then $||x - x_f||^2 \leq |x|^2 (\frac{\epsilon}{\lambda})^2 |\lambda|^2$. Therefore we obtain $||x - x_f||^2 < \epsilon ||x||^2$ and finally we find the resulting equation $||x - x_f|| < \epsilon ||x||$.

5. Conclusion

I would like to continue in this research and eventually use a computer program such as MathLab ot M.A.P.L.E. to solve Linear equations of Fredholm type of finite rank. After I understand the proofs I could possibly use this research and apply it to Fuzzy numbers.

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Fractional Differential Equations

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Abstract Using Caputo's fractional differential equation, we develop numerical methods. In particular for Caputo's fractional differential equation, we develop the Improved Euler method for fractional differential equations.

1. Introduction

The concept of fractional calculus was originally proposed by famous French mathematician Guillaume de L'Hopital. He asked fellow mathematician Gottfried Wilhelm Leibniz if it was possible if the n^{th} derivative of a function could be equal to $\frac{1}{2}$. Gottfried Wilhelm Leibniz was initially unsure if such a thing was possible, but not long after that the idea caught attention to other mathematicians who have contributed to fractional derivatives in some kind of way. The mathematicians that have contributed to fractional derivatives include: Bernhard Riemann and Joseph Liouville, Anton Karl Grünwald and Alekesy Vasilievich Letnikov, and M. Caputo. All these well known mathematicians ended up creating their own type of fractional derivative. In the next section we include some preliminaries. In Section 3 we define various types of fractional derivatives In Section 4 we have presented Improved Euler Method.

2. Preliminaries

Fractional derivatives takes place when the n^{th} derivative of a function is a fraction instead of an integer. To find the derivative for fractional derivatives, we will be utilizing what is known as the Gamma function. The Gamma function is defined as $\Gamma(x) = (x-1)!$ and it is used when we take the fractional derivative of a function. An example to illustrate this would be: $y = x^m$, where m is positive. The notation for the fractional derivative is written as:

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, m \geq n. \text{ [?]}$$

Using the Gamma function we are able to rewrite our fractional derivative as $\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}$.

As an example, let $f(x) = x$. In this function, m equals 1. Since the n^{th} derivative equals $\frac{1}{2}$ we calculate the derivative which results in:

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-\frac{1}{2}} = \frac{(1)!}{(1-\frac{1}{2})!} x^{1-\frac{1}{2}} = \frac{\sqrt{x}}{\frac{\sqrt{\pi}}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

Here we discovered that we are able to find the factorial of fraction, where in the past we usually deal with integers. The main difference in factorials of fractions is that the end result is multiplied by $\sqrt{\pi}$. Now let's see what happens when we take the $\frac{1}{2}^{th}$ derivative of our result $\frac{2\sqrt{x}}{\sqrt{\pi}}$.

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{(\frac{1}{2})!}{(\frac{1}{2}-\frac{1}{2})!}x^{\frac{1}{2}-\frac{1}{2}} * \frac{2}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{2}x^0(\frac{2}{\sqrt{\pi}}) = 1.$$

This shows that when we take $\frac{1}{2}^{th}$ of a function twice, the result will be the same as taking the first order derivative of the same function. Here is a table of $f(x)$ functions and their $\frac{1}{2}^{th}$ derivatives.

TABLE 1. Other Fractional Derivatives

$f(x)$	$\frac{d^{\frac{1}{2}}f}{dx^{\frac{1}{2}}}$
0	0
C (constant)	$\frac{C}{\sqrt{\pi x}}$
1	$\frac{1}{\sqrt{\pi x}}$
x	$2\sqrt{\frac{x}{\pi}}$
x^2	$\frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}}$
$x^{\frac{1}{2}}$	$\frac{\sqrt{\pi}}{2}$

However, understanding this basic definition isn't enough to understand fractional calculus, as there other types of fractional derivatives being studied in our research.

3. Types of Fractional Derivatives

As mentioned earlier, L'Hopital's constant questioning of the possibility of a $\frac{1}{2}^{th}$ derivative got the attention of other mathematicians, which resulted in different formulas of fractional derivatives being created other the years. The three types that are focused on in this paper are Gr*ü*nwald-Letnikov fractional derivative, Riemann-Liouville fractional derivative, and Caputo's fractional derivative.

Gr*ü*nwald-Letnikov fractional derivative is considered to be an extension of the derivative formula in terms of fractional calculus. Using this formula will enable us to take the derivative of a function a fractional number of times. The formula begins with the first-derivative function defined as: $f'(x) = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$. We then proceed to take the first derivative of the function again leading to the second derivative function which is:

$$\begin{aligned} f''(x) &= \lim_{h\to 0}\frac{f'(x)-f'(x-h)}{h} \\ &= \lim_{h\to 0}\frac{1}{h} * (\frac{f(x)-f(x-h)}{h} - \frac{f(x-h)-f(x-2h)}{h}) \\ &= \lim_{h\to 0}\frac{f(x)-2f(x-h)-f(x-2h)}{h^2}. \end{aligned}$$

It is through induction that the formula for the n^{th} derivative is defined as:

$f^{(n)}(x) = \frac{d^n f}{dx^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r \binom{n}{r} f(x-rh)$, where $\binom{n}{r} = \frac{n(n-1)(n-2).....(n-r+1)}{r!}$.

We know that n represents integers we use for derivatives, so we will use α to represent our fractional derivatives. This allows us to rewrite the formula as: $f^{(\alpha)}(x) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{r=0}^{n}(-1)^r \binom{\alpha}{r} f(x-rh)$. The final formula for fractional derivatives is given as:

$$ {}_0D_x^{\frac{1}{2}} f(x) = \sum_{k=0}^{n} \frac{f'(a)(x-a)^{-\frac{1}{2}+1}}{\Gamma(-\frac{1}{2}+1+1)} + \frac{1}{\Gamma(-\frac{1}{2}+1+1)} \int_a^x (x-\tau)^{1-\frac{1}{2}} f^{1+1}(\tau) d(\tau),\ n < \alpha < n+1. \tag{3.1} $$

The second fractional derivative studied was the Riemann-Liouville fractional derivative. This derivative is used when we are dealing with non-integral terms in a function. The formula is defined as

$$ {}_aD_x^{\alpha} f(x) = (\tfrac{d}{dx})^{n+1} \int_a^x (x-\tau)^{n-\alpha} f^{n+1}(\tau) d(\tau),\ n < \alpha < n+1. \tag{3.2} $$

The fractional derivative can be reinterpreted as

$$ \begin{aligned} {}_aD_x^{\alpha} f(x) &= (\frac{d}{dx})^{n+1} \int_a^x (x-\tau)^{n-\alpha} f^{n+1}(\tau) d(\tau) \\ &=_0 D_x^{\frac{1}{2}} f(x) = \sum_{k=0}^{n} \frac{f'(a)(x-a)^{-\frac{1}{2}+1}}{\Gamma(-\frac{1}{2}+1+1)} + \frac{1}{\Gamma(-\frac{1}{2}+1+1)} \int_a^x (x-\tau)^{1-\frac{1}{2}} f^{1+1}(\tau) d(\tau). \end{aligned} $$

It is shown here that the Riemann-Liouville fractional derivative is a condensed version of the Grünwald-Letnikov fractional derivative. The main difference is in the Riemann-Liouville derivative the derivative, $\frac{d}{dt}$ is left out side the integral while for the Grünwald-Letnikov derivative the derivative is inside the integral.

Caputo's fractional derivative differs from the Riemann-Liouville fractional derivative as it is not required to define the initial conditions of the fractional order. While the Riemann-Liouville fractional derivative provides an important role in fractional derivatives' definition and application to pure mathematics, revised version of the pure math approach is required. Caputo's fractional derivative is defined as

$$ {}_a^C D_x^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(\tau)}{(x-\tau)^{\alpha+1-n}} d(\tau),\ n-1 < \alpha < n. \tag{3.3} $$

Here as $\alpha \to n$, the Caputo derivative becomes a conventional n^{th} derivative of $f(t)$. We assume that $0 \le n-1 < \alpha < n$ and that the function f(t) has $n+1$ continuous bounded derivatives in [a, T] $\forall\ T > a$.

Now we will do a comparison between Grünwald-Letnikov and Caputo fractional derivative since Riemann-Liouville is a condensed version of Grünwald-Letnikov. The $f(x)$ used will be $f(x) = x$.

Grünwald − Letnikov

$$ \begin{aligned} {}_0D_x^{\frac{1}{2}} f(x) &= \sum_{k=0}^{n} \frac{f'(a)(x-a)^{-\frac{1}{2}+1}}{\Gamma(-\frac{1}{2}+1+1)} + \frac{1}{\Gamma(-\frac{1}{2}+1+1)} \int_a^x (x-\tau)^{1-\frac{1}{2}} f^{1+1}(\tau) d(\tau) \\ &= \sum_{k=0}^{n} \frac{1 * (x-a)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{3}{2})} \int_a^x (x-\tau)^{\frac{1}{2}} f^2(\tau) d(\tau) \\ &= \sum_{k=0}^{n} \frac{(x-a)^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{1}{(\frac{1}{2})!} \int_a^x (x-\tau)^{\frac{1}{2}} * 0 d(\tau) \\ &= \sum_{k=0}^{n} \frac{(x-a)^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2}} + 0 \\ &= \sum_{k=0}^{n} \frac{2\sqrt{x-a}}{\sqrt{\pi}}. \end{aligned} $$

Since $a = 0$, ${}_0D_x^{\frac{1}{2}} f(x) = 2\frac{\sqrt{x}}{\sqrt{\pi}}$.

Caputo

$$\begin{aligned}
{}_a^C D_x^\alpha f(x) &= \frac{(-1)^1}{\Gamma(1-\frac{1}{2})} \int_a^x \frac{f'(\tau)}{(x-\tau)^{\frac{1}{2}+1-1}} d(\tau) \\
&= \frac{-1}{\Gamma(\frac{1}{2})} \int_a^x \frac{1}{(x-\tau)^{\frac{1}{2}}} d(\tau) \\
&= \frac{-1}{\frac{-1}{2}!} \int_a^x (x-\tau)^{\frac{-1}{2}} d(\tau) \\
&= \frac{-1}{\sqrt{\pi}} [2(x-\tau)^{\frac{1}{2}}]|_a^x \\
&= \frac{-1}{\sqrt{\pi}} [2\sqrt{x-x} - 2\sqrt{x-a}] = 2\frac{\sqrt{x-a}}{\sqrt{\pi}}.
\end{aligned}$$

Since $a = 0$, ${}_a^C D_x^\alpha f(x) = 2\frac{\sqrt{x}}{\sqrt{\pi}}$.
This comparison shows that both (3.1) and (3.3) give us the same result, but (3.3) found the result in an easier way. Now that we understand the three different types of fractional derivatives, now we can begin our theorems on Euler's and improved Euler's method.

4. Euler/Improved Euler Method

First we state the Taylor Series Expansion. Theorem 1, Taylor Series Expansion: Based on our understanding of series expansion and differential equations, show that the fractional Taylor power series is defined as:

$$f(x) = \sum_{j=0}^{n} {}_cD_a^{j\alpha} f(a_1) \frac{\Delta_j}{\Gamma(j\alpha+1)} + R_n(x, a_1, a).$$

Proof: The method begins by letting $\alpha \in (0, 1]$, thus making it a fractional order. We also let f(x) be a continuous function in [a, b], which satisfy the following conditions:

1. $\forall j = 1, 2, ..., {}_cD_a^{j\alpha} f \in C([a, b]), {}_aI_\alpha([a, b])$.
2. ${}_cD_a^{9\alpha} f(x)$ is continuous on [a, b].

Let $x = a_1$ for the function such that $a_1 \in [a, b]$. Then $\forall x \in [a, b]$,

$$\begin{aligned}
f(x) = f(a_1) + {}_cD_a^\alpha f(a_1) \frac{\Delta_1}{\Gamma(\alpha+1)} + {}_cD_a^{2\alpha} f(a_1) \frac{\Delta_2}{\Gamma(2\alpha+1)} + {}_cD_a^{3\alpha} f(a_1) \frac{\Delta_3}{\Gamma(3\alpha+1)} \\
+ {}_cD_a^{4\alpha} f(a_1) \frac{\Delta_4}{\Gamma(4\alpha+1)} + R_4(x, a_1, a),
\end{aligned}$$

with $\Delta_1, \Delta_2, \Delta_3$, and Δ_4 being the differences given by:
$\Delta_1 = [H^\alpha - L^\alpha]$,
$\Delta_2 = [H^{2\alpha} - L^{2\alpha} - \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} L^\alpha \Delta_1]$,
$\Delta_3 = [H^{3\alpha} - L^{3\alpha} - \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)} L^\alpha \Delta_2 - \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)\Gamma(\alpha+1)} L^{2\alpha} \Delta_1]$,
$\Delta_4 = [H^{4\alpha} - L^{4\alpha} - \frac{\Gamma(4\alpha+1)}{\Gamma(\alpha+1)\Gamma(3\alpha+1)} L^\alpha \Delta_3 - \frac{\Gamma(4\alpha+1)}{\Gamma^2(2\alpha+1)} L^{2\alpha} \Delta_2 - \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+1)} L^{3\alpha} \Delta_1]$.
Here, $H = (x - a)$ and $L = (a_1 - a)$. The remainder term is $R_4(x, a_1, a)$.

Now that we understand fractional Taylor series, we will begin writing out Euler's method as a fractional derivative.

In order to write out the fractional Euler's method, we will use the fractional Taylor series to do so. Instead of using and ODE, we will be using a fractional differential equation (FDE).

Theorem 2 (Fractional Euler Method): Let,

$$\text{FDE: } {}_cD_0^\alpha y(t) = f(y),\ y(0) = y_0 \tag{4.1}$$

For the FDE (4.1) to show that the **Fractional Euler Method** is

$$w_{i+1} = w_i + \tfrac{f(w_i)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha).$$

Proof: Let $y(t)$ be the solution of (4.1) for some $t = t_i$, where $i = 0, 1, 2...n$. We first solve for $y(t_1)$ where $i = 0$. This leads to the function being $y(t_1) = y(t_0) + {}_cD_0^\alpha y(t_0)\frac{\Delta_1}{\Gamma(\alpha+1)} + R_1(t, t_0, 0)$. Now we analyze $y(t_{1+1}) = y(t_2)$, where $i = 1$.

$$y(t_2) = y(t_1) + {}_cD_0^\alpha y(t_1)\tfrac{\Delta_1}{\Gamma(\alpha+1)} + R_1(t, t_1, 0).$$

Based on fractional Taylor series the equation is written as

$$y(t_{i+1}) = y(t_i) + {}_cD_0^\alpha y(t_i)\tfrac{\Delta_1}{\Gamma(\alpha+1)} + R_1(t, t_i, 0), \text{ where } \Delta_1 = h^\alpha((i+1)^\alpha - i^\alpha).$$

Now we us the FDE of (4.1) and Δ_1 which leads to the function being $y(t_{i+1}) = y(t_i) + \frac{f(y(t_i))h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha) + R_1(t, t_i, 0)$.

Just like in Euler's method, there exists an approximation in the fractional Euler method. From the function $y(t)$ at t_i, the approximation is defined as w_i, where $w_i \approx y(t_i)$. This leads to the final formula being

$$w_{i+1} = w_i + \tfrac{f(w_i)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha), \text{ where } w_0 = y_0.$$

Therefore, we have shown that $w_{i+1} = w_i + \frac{f(w_i)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. (4.2)

Now that we have written out Euler's method for fractional differential equations, we will do the same with improved Euler's method.

Writing improved Euler's method will be similar to writing an extension of the fractional Euler's method.

Theorem 3 (Fractional Improved Euler Method):

For the FDE (4.1) and with the FEM (Fraction Euler Method) (4.2), show that the general formula for the fractional improved Euler method is

$$w_{i+1} = w_i + \tfrac{(f(w_i)+f(w_{i+1}^*))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha).$$

Proof: We proceed in the same manner to for this method as we did with the improved Euler method. Here the function $w_i \approx y(t_i)$. Let the fractional Euler method be $w_{i+1}^* = w_i + \frac{f(w_i)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. Fractional Euler method must be computed first and is labeled with a "*" so as not to be mixed up with w_{i+1}, which is used to label the improved fractional Euler method. Hence the fractional Euler method is introduced as $f(w_{i+1}^*)$. The result of w_{i+1}^* will be added with $f(w_i)$. We know the sum will be multiplied by $\Delta_1 = h^\alpha((i+1)^\alpha - i^\alpha)$ and that the sum, $y_{n+1} = y_n + \frac{h(f(x_n,y_n)+f(x_{n+1},y_{n+1}^*))}{2}$, is divided by two. Since we are using fractional derivatives, $\frac{1}{2}$ must have an α exponent as well. This leads to $\Delta_1 = (\frac{h}{2})^\alpha((i+1)^\alpha - i^\alpha)$.

When we solve for w_1^* when $i = 0$ it results in $w_1^* = w_0 + \frac{f(w_0)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. Then the fractional improved Euler method is defined as

$$w_1 = w_0 + \tfrac{(f(w_0)+f(w_1^*))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha).$$

Then the next iteration is when $i = 1$: $w^*_{1+1} = w^*_2 = w_1 + \frac{f(w_1)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. Then we substitute w^*_2 in the formula.

$$w_2 = w_1 + \frac{(f(w_1)+f(w^*_2))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha).$$

Similarly the iteration fo $i = n-1$ is $w^*_n = w_{n-1} + \frac{f(w_{n-1})h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$.

$$\text{This leads to } w_n = w_{n-1} + \frac{(f(w_{n-1})+f(w^*_n))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$$

The last iteration is for $i = n$. Here $w^*_{n+1} = w_n + \frac{f(w_n)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. This results in the fractional improved Euler method being

$$\begin{aligned} w_{n+1} &= w_n + \frac{(f(w_n) + f(w_n + \frac{f(w_n)h^\alpha}{\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha) \\ &= w_n + \frac{(f(w_n) + f(w^*_{n+1}))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha). \end{aligned}$$

Therefore, by iteration, we have shown that

$w_{i+1} = w_i + \frac{(f(w_i)+f(w^*_{i+1}))h^\alpha}{2^\alpha\Gamma(\alpha+1)}((i+1)^\alpha - i^\alpha)$. (4.3)

5. Conclusion

We developed the fractional improved Euler method after understanding the basic concept of fractional derivatives. Now, both fractional Euler's method and fractional improved Euler's method can be used as numerical solutions to fractional differential equations.

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A Mathematical Model for Solving Jigsaw Sudoku Puzzles

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Abstract The Sudoku puzzle is the composite result of the study of magic squares, Latin squares and gerechte designs over multiple centuries. We provide a brief history of this study, as well as an outline of the rules of different variations of the puzzle, including both traditional and Jigsaw Sudoku. We explain how Arnold, Lucas, and Taalman showed that the Gröbner basis can be used to characterize the solution of a traditional Sudoku puzzle. Bartlett and Langville created an integer programming model that provides a solution to a traditional Sudoku puzzle by mathematically implementing the rules of the puzzle. By modifying a portion of this model to conform to the rules of Jigsaw Sudoku, we produce solutions to Jigsaw Sudoku puzzles.

1. The History Behind Sudoku

Before one can properly understand the research conducted, it is necessary to examine the history of the Sudoku puzzle. The origins of Sudoku date back to ancient Asian and Middle Eastern civilizations' interest in magic squares. We define a *magic square* as an $n \times n$ square containing the numbers 1 through n^2, denoted $1 : n^2$, such that all rows, columns, and diagonals have a constant sum. For example, Figure 1 is a magic square in which all rows, columns, and diagonals have a sum of 15. In the 18th century, Leonhard Euler popularized the idea of the magic square in Europe [**5**].

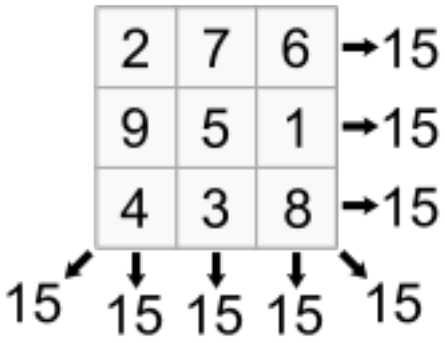
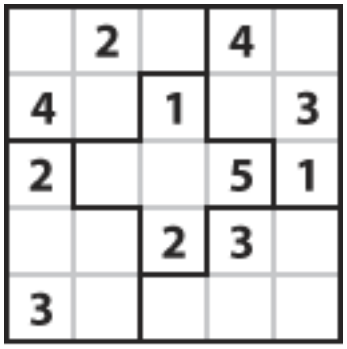


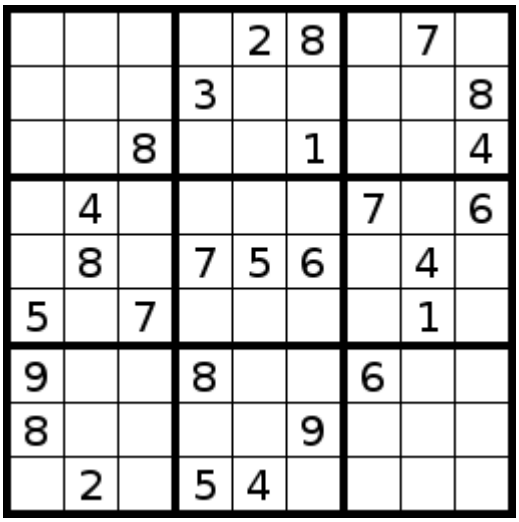
Figure 1

Two centuries later, Ronald A. Fisher manipulated the magic square into the *Latin square*, an $n \times n$ array that contains the numbers $1 : n$ only once in each row and column. In 1950, W. U. Behrens invented *gerechte designs*, the division of an $n \times n$ grid into n regions of n cells. Figure 2 illustrates a possible gerechte design for a 5×5 grid [**2, 5**].

Figure 2 

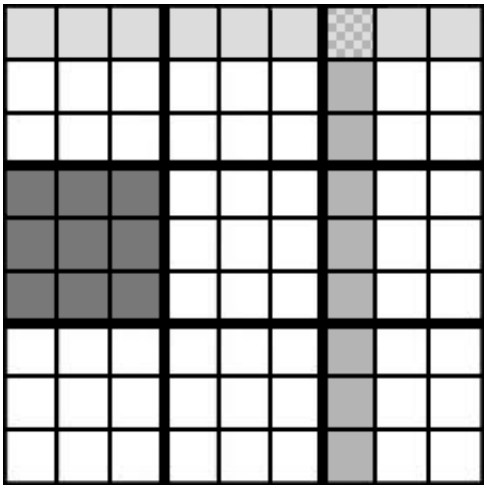
John Nelder placed more restrictions on the Latin square in 1977. He called a *critical set* a portion of the Latin square that can only be completed in one way. He also defined a *trade* as a collection of entries which can be substituted for different entries to create a different Latin square.

In 1979, Howard Garns combined these developments and created the *Number Place* puzzle, a critical set of a gerechte design of a 9×9 grid partitioned into 3×3 subsquares. We call these subsquares *submatrices*. Number Place migrated to Japan, where the Japanese renamed it *Sudoku*, an abbreviation for the phrase *suji wa dokushin ni kagiru*, or "the digits must remain single." Entrepreneur Wayne Gould found Sudoku interesting and recognized its potential to sell. In the 1990s he created a computer program to generate unique puzzles, like the one pictured in Figure 3, at amazing rates. The beginning of the 21st century marked the boom of the Sudoku puzzle in the United States of America [**2, 3, 4, 5**].

Figure 3 

2. Sudoku Specifics

The Sudoku puzzle is of size $n \times n$, where $n = m^2$, and m is any positive integer. Every row, column, and $m \times m$ submatrix contains the digits $1 : n$ exactly once. Figure 4 identifies a row (light gray), a column (medium gray) and a submatrix (dark gray). Note that every row and column intersect in one cell somewhere in the puzzle. In addition, no $m \times m$ submatrix overlaps another. The darker boundary lines distinguish the cells of one submatrix from the cells of another. These constraints form the basic rules of any Sudoku game [**3, 4**].

Figure 4 

Sudoku evolved throughout the years into numerous variations, based on either size or additional constraints. Four common sizes of the Sudoku puzzle exist: the 4×4 *Shidoku* puzzle, Garns's original 9×9 *Sudoku* puzzle, the 16×16 puzzle, and the 25×25 *Samurai Sudoku* puzzle. Each of these follows the basic rules of Sudoku. Many Sudoku puzzle variations include additional constraints, including Snowflake, Argyle, Four Squares, Pyramids, Stairs, Staples, Size 8, Holes, Worms, Color, and Jigsaw Sudoku [**6, 7**]. Many of these are quite similar to each other, so we outline the constraints for just a select few.

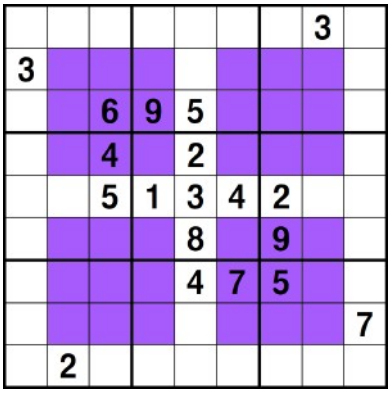
Argyle Sudoku (Figure 5) follows the basic rules, but includes diagonals in which repeating entries do not exist. Note that not every integer $1 : n$ appears in the diagonals, because each diagonal covers less than n cells [**6**].

Figure 5

		2						3
	6							
1	3	5		4				
	8	1	5					
		3				9		
					1	5	2	
				1		8	3	7
							9	
4						6		

Figure 6 illustrates a Four Squares Sudoku puzzle. Four Squares Sudoku retains the basic rules, but includes four extra colored subsquares in which the integers $1 : n$ are allowed exactly once. The colored subsquares overlap the submatrices, but do not overlap each other [**6**].

Figure 6



Finally, we consider the *Jigsaw Sudoku* puzzle. The Jigsaw variation retains the basic constraints that require the integers $1:n$ in each row and column exactly once, but discards the submatrix constraint altogether. Instead of submatrices, Jigsaw Sudoku harbors n jigsaw pieces composed of n adjacent cells and arranged in any way such that all n Jigsaw pieces will fit in the $n \times n$ square. As illustrated in Figure 8, the light gray cells are *adjacent* to the dark gray cells because the they are vertically or horizontally side-by-side. In Jigsaw Sudoku, each jigsaw piece contains $1:n$ exactly once [**6**].

Figure 7

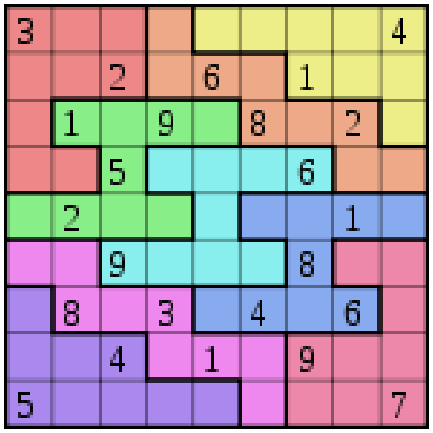
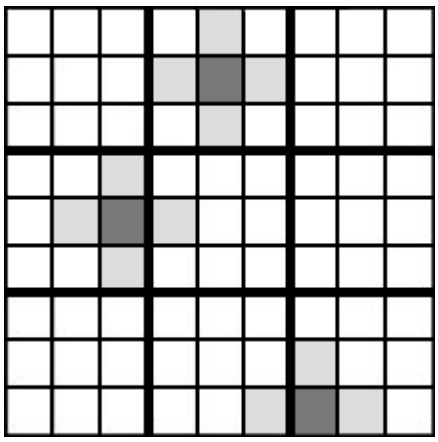


Figure 8



In [**1**], Arnold, Lucas and Taalman showed how Gröbner bases can identify if Sudoku puzzles have one, many, or no solutions. They also outlined how to use the Gröbner basis to solve an actual puzzle. Bartlett and Langville formulated a mathematical model in MatLab called *sudoku.m* to quickly solve a Sudoku puzzle with basic constraints [**3, 4**]. Little work exists on Jigsaw Sudoku so it became the focus of our research. We chose to modify Bartlett and Langville's *sudoku.m* file model to solve a Jigsaw puzzle and to focus on Shidoku-sized puzzles for purposes of simplicity.

3. Gröbner Basis

We can represent a Sudoku board of any size as a system of polynomials by assigning a variable to each cell of the board. The *sum-product system* uses the equations for the sum and product of each row, column, and submatrix to create a polynomial system.

Consider the 4×4 Shidoku board with basic constraints. Assigning each cell a variable gives a set of sixteen variables $\{a, b, c, \ldots, p\}$. By the rules of the game, each variable falls inclusively between 1 and n, with $n = 4$ in this case. Thus, we have sixteen equations

of the form $(a-1)(a-2)(a-3)(a-4) = 0$ that guarantee each cell's value is in this range. We also know the sum of each row, column and submatrix. For Shidoku, adding the variables together in each row, column and submatrix of the board results in a 10 every time. This produces twelve equations of the form $1+2+3+4-10 = 0$: four for the rows, four for the columns, and four for the submatrices. These equations guarantee that no other combination of the four integers that sum to 10, for example $\{2,2,3,3\}$, is used. Similarly, multiplying the variables together in each row, column, and submatrix of the board results in a 24 every time. Thus, there are another twelve equations of the form $(1)(2)(3)(4) - 24 = 0$, one for each row, column, and submatrix to guarantee that no other combination of the four integers with a product of 24, for example $\{2,2,2,3\}$ is used. Taken together, this produces a foundational system of forty polynomial equations for any Shidoku board that ensures each integer is only used once in each row, column, and submatrix.

While solving the system proves tedious and complicated, finding the equations of the sum-product system does not. As defined in [**1**], a *Gröbner basis* for a system of polynomials is a new system of polynomials with the same solutions as the original system, but which is easier to solve. Including polynomials to represent the given values in the system allows the Gröbner basis to accomplish two key goals: to determine whether or not the puzzle is solvable and, if it is solvable, to identify the solution board.

In addition to the forty foundational equations, we include others to represent any integer values given in the puzzle. For example, if the value in cell a is 2 and the value in cell d is 4, as in Figure 9, then we add the equations $a-2=0$ and $d-4=0$ to the system.

Figure 9

2			4
		1	
	3		

The number of given values included in the system plays a role in what the Gröbner basis can explain. When we do not add any additional equations to the system to represent the given values, the Gröbner basis will return a set of equations that will simply model the structure of the board. When we add an inconsistent set of given equations to the system, the Gröbner command in Maple returns the polynomial 1, which indicates that $1 = 0$, an impossible occurrence. Thus no possible solution exists for the puzzle. The Gröbner basis consists of some solutions and some polynomials if the system remains undetermined. This occurs when we do not add enough given equations to guarantee a unique solution. Finally, and hopefully this is the case, the Gröbner command returns linear polynomials of the form $a-w$, indicating $a-w=0$, where w is an integer within our range. Simple calculations return the values for each variable and identify the solution board.

The Gröbner command proves rather complicated, especially in our case with so many multivariable polynomials. If our system contains only linear polynomials, then the Gröbner basis is the system of equations taken from the reduced echelon form of the matrix of polynomial equations. When the system consists of only equations with one variable, the Gröbner basis equals the greatest common divisor of the system. We find this using a series

of reduction steps. Since the sum-product Shidoku system is multivariable and nonlinear, MatLab uses a combination of the two methods to determine the Gröbner basis.

We can also find a Gröbner basis by using the ideal method as outlined in [**1**]. An *ideal* is a subset of a polynomial ring that is closed under polynomial addition and closed under multiplication of all polynomials in the ring. The *leading term* of any polynomial is simply the first term (coefficient and variable) in the polynomial when the terms of the polynomial are ordered lexicographically. Let T be a set of polynomials and I be the ideal generated by T. To find the leading term ideal of T, $Lt(T)$, the polynomials of T are simplified to just their leading term, and the ideal generated by those new polynomials is found. Similiarly, one can find the leading term ideal of I, $Lt(I)$. When $Lt(T) = Lt(I)$, T is a Gröbner basis for I [**1**].

4. Integer Programming Model

In [**3**] and [**4**], Bartlett and Langville supplied an integer programming model, *sudoku.m*, that solves any Sudoku puzzle. Bartlett and Langville used MatLab to implement the constraints for the puzzles. To begin, the size of the puzzle is specified by setting n equal to the square of an integer. They called i the row number, j the column number, and k the integer value. The decision variable is defined to be

$$x_{ijk} = \begin{cases} 1 & \text{if element } (i,j) \text{ of the } n \times n \text{ Sudoku matrix contains the integer } k \\ 0 & \text{otherwise} \end{cases}$$

for the general $n \times n$ puzzle. The vector $\mathbf{x}$ of dimension $(n^3, 1)$, contains all of the decision variables. The matrix **Aeq** represents the $(4n^2, n^3)$ matrix of equations calculated from the values of i, j, and k. Originally, **Aeq** is a zero matrix. The *sudoku.m* programs inserts ones into **Aeq** according to summation equations that ensure the following constraints are met:

1) only one k in each column
2) only one k in each row
3) only one k in each submatrix
4) every position in the puzzle is filled, and
5) all of the given values are included.

There are n^2 total equations for each of the first four constraints and r equations for the fifth constraint, where r is the number of given elements for the puzzle.

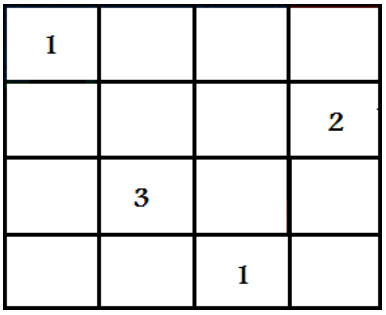


Figure 10

For the purposes of this example, assume that we are working with the 4×4 Shidoku puzzle with basic rules shown in Figure 10. In order to be compliant with the rules, each column has only one of each integer $1 : n$. We represent the columns constraint for a Shidoku puzzle in **Aeq** by the rows 1 through 16. The second constraint allows only one of each

integer $1:n$ for each row. We represent the rows constraint for Shidoku in **Aeq** by rows 17 through 32. The third constraint represents that only one integer $1:n$ can be in each submatrix and yields rows 33 through 48 in **Aeq**. Rows 49 through 64 of **Aeq** ensure that the fourth constraint applies. We represent the givens constraint by rows 65 through $65+r$ of the **Aeq** matrix, where r is equal to the total number of givens.

Inputting the givens requires a different approach from the equations used in constraints 1 through 4. Bartlett and Langville created a matrix **G** of dimension $(r, 3)$, where each row of **G** identifies the position and value of the puzzle's given values. Each row of **G** is a $(1, 3)$ matrix of the form $[i, j, k]$, with i identifying the row number, j the column number, and k the value of the integer. For example, the matrix **G** for Figure 10 is

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 2 & 3 \\ 4 & 3 & 1 \end{bmatrix}.$$

In order to transform our large **Aeq** matrix into a Shidoku board, *sudoku.m* uses MatLab to multiply the updated **Aeq** by the decision variable, **x**, and produces the matrix **beq**, a $(4n^2 + r, 1)$ matrix of ones. **Beq** instructs MatLab to turn **x** into Shidoku terms by methodically searching through each constraint's rows for the positions of nonzero elements of **x**. Those positions select the needed decision variables, which identify the (i, j) positions of each k's locations in the Shidoku board. [**3, 4**].

We made modifications to the *sudoku.m* file that allowed us to produce solutions for Jigsaw Shidoku puzzles. The only difference between a Jigsaw Sudoku puzzle and an original Sudoku puzzle is one constraint. We eliminated the submatrix constraint completely and replaced it with a constraint requiring that the integers $1:n$ appear exactly once in each of the n jigsaw pieces. In order to do this we labeled the shapes as pieces using a method similar to the givens constraint, inputting a letter label in the third column rather than an integer value.

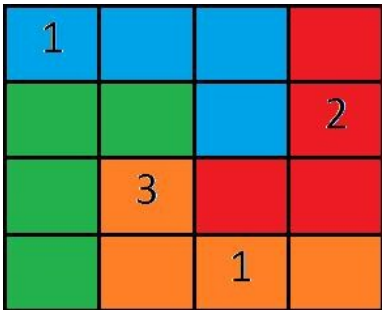


Figure 11

For example, in Figure 11, we call the jigsaw pieces (clockwise from upper left) A, B, C, and D. Our Pieces matrix is of dimension $(n^2, 3)$ and appears below.

$$\mathbf{Pieces} = \begin{bmatrix} 1\ 1\ A \\ 1\ 2\ A \\ 1\ 3\ A \\ 2\ 3\ A \\ 1\ 4\ B \\ 2\ 4\ B \\ \vdots \\ 2\ 1\ D \\ 2\ 2\ D \\ 3\ 1\ D \\ 4\ 1\ D \end{bmatrix}$$

This method requires the definition of all jigsaw pieces in order to solve the puzzle. We use a different letter for each n colored pieces. Each letter will take on each of the values within our range $1 : n$ only once, guaranteeing that we have only one of every value in each jigsaw piece. Once we do this, the final output of the modified *sudoku.m* file yields the $n \times n$ **S** matrix, a solution to the puzzle.

$$S = \begin{bmatrix} 1\ 4\ 2\ 3 \\ 4\ 1\ 3\ 2 \\ 2\ 3\ 4\ 1 \\ 3\ 2\ 1\ 4 \end{bmatrix}$$

5. Results and Future Study

Through our research we thoroughly understood the mathematics behind the Gröbner basis. We deciphered the *sudoku.m* file and analyzed step-by-step how MatLab solved the Sudoku and Shidoku puzzles. Once we processed the characteristics of the program by hand, we proceeded to do the same with our modified constraints. The restructured *sudoku.m* file obtained an accurate solution for Jigsaw Shidoku puzzles and we manually understood and produced a solution for Jigsaw Sudoku puzzles.

During our research we concluded that our results could be extended in numerous directions. Additional work could efficiently calculate the Gröbner basis for any set of polynomials to characterize the solution to a specific puzzle. Further research could modify the original *sudoku.m* file to produce the solutions for other variations and sizes of Sudoku puzzles. The *sudoku.m* file might also help to find a unique solution for an original 9×9 puzzle with 16 givens, as [**1**] claims that 17 is the smallest possible number of givens that has been found to identify a unique solution.

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The zero-forcing numbers of signed graphs

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Abstract: In this paper we introduce the zero-forcing number of a signed graph. The motivation for introducing the zero-forcing number of a signed graph comes from the computation of the maximum nullity of a signed graph. For some classes of signed graphs, we determine the zero-forcing number.

1. Introduction

A graph G consists of a finite set V of vertices and a finite set E of pairs of distinct vertices, called edges. If v and w are two vertices that belong to an edge e, we say that e connects v and w; we will also say that v and w are *adjacent*. A *cycle* in G is a sequence of alternating distinct vertices and edges $v_1, e_1, v_2, \ldots, v_k$ such that e_1 connects v_1 and v_2, e_2 connects v_2 and v_3, ..., e_{k-1} connects v_{k-1} and v_k, and an edge connects v_k and v_1.

If G is a graph with vertex-set $\{1, 2, \ldots, n\}$, we define $S(G)$ to be the set of all real symmetric $n \times n$ matrices $A = [a_{i,j}]$ such that

(1) $a_{i,j} \neq 0$ if $i \neq j$ and i and j are adjacent vertices,
(2) $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent vertices,
(3) $a_{i,i} \in \mathbb{R}$ for $i = 1, \ldots, n$.

For example, the matrix

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 3 & 1 \\ -1 & 3 & -2 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

belongs to $S(K_4)$.

The maximum nullity, $M(G)$, of a graph G is the maximum of all nullities of matrices $A \in S(G)$. For some graphs G, the maximum nullity $M(G)$ has been determined, see [3] for a survey. But in general determining the maximum nullity of a graph is difficult. Obtaining a lower bound for the maximum nullity of a graph is easy: the nullity of any matrix $A \in S(G)$ provides a lower bound for the maximum nullity of a graph. But how can we obtain an upper bound for the maximum nullity of a graph?

In [1], the following graph parameter was introduced, which provides an upper bound for the maximum nullity of a graph, and thus helps in determining the maximum nullity of a graph. Let $G = (V, E)$ be a graph. For a subset S of V, color the vertices in S black and color the other vertices white. We now repeatedly apply the following rule. If a black vertex has all but one of its neighbors black, color the white vertex black. We might end with having all vertices black. If this happens, then we say that S is a *zero-forcing set*. The *zero-forcing number* of G is the smallest size of a zero-forcing set. In [1], it was shown that the zero-forcing number of a graph G forms an upper bound for the maximum nullity of G.

For every graph G, $M(G) \leq Z(G)$.

The *complete graph* K_n is the graph on n vertices and each pair of distinct vertices is connected by an edge. The *complete bipartite graph* $K_{p,q}$ is the graph with $p+q$ vertices which has a partition into two parts V_1 and V_2, where V_1 contains p vertices and V_2 contains q vertices, such that each vertex in V_1 and each vertex in V_2 are connected by an edge, and these are the only edges.

2. The zero-forcing number of a signed graph

A *signed graph* is a pair (G,ϕ), where $G=(V,E)$ is a graph and $\phi : E \to \{-,+\}$ is a function which assigns to each edge a $+$ or a $-$. We call the edges e with $\phi(e)=+$ positive edges and the other edges negative edges. We call a cycle C in a signed graph positive if C has an even number of negative edges. (Think about multiplying the signs of all edges in the cycle.) A *signing* of a graph $G=(V,E)$ is a signed graph (G,ϕ), where ϕ is a function $\phi : E \to \{-,+\}$. If U is a subset of the vertex set of G, then *switching along* U means that we swap the signs of all edges in $\delta(U)$. If (G,ϕ) and (G,ψ) are signed graphs, then we say that (G,ϕ) is sign-equivalent to (G,ψ) if (G,ϕ) can be obtained from (G,ψ) by switching along a subset of the vertex set of G. Zaslavsky [4] showed that (G,ϕ) and (G,ψ) are sign-equivalent if and only if they have the same set of positive cycles.

If (G,ϕ) is a signed graph with vertex-set $\{1,2,\ldots,n\}$, we define $S(G,\phi)$ to be the set of all real symmetric $n\times n$ matrices $A=[a_{i,j}]$ such that

(1) $a_{i,j}<0$ if $i\neq j$ and the edge between i and j is negative,
(2) $a_{i,j}>0$ if $i\neq j$ and the edge between i and j is positive,
(3) $a_{i,j}=0$ if $i\neq j$ and i and j are not adjacent,
(4) $a_{i,i}\in\mathbb{R}$ for $i=1,\ldots,n$.

In [2], the maximum nullity, $M(G,\phi)$, of a signed graph (G,ϕ) was introduced. It is defined to be the maximum of all nullities of matrices $A\in S(G,\phi)$, and it is invariant under sign-equivalence. For some signed graphs, the maximum nullity has been determined, but in general it is difficult to find. In this paper, we introduce the *zero-forcing number* of a signed graph.

If U and W are mutually disjoint subsets of the vertex set of a signed graph (G,ϕ), then the $\{+,-,0\}$-*sign pattern matrix* on $U\times W$ is the $U\times W$ matrix $A=[a_{i,j}]$ with $a_{i,j}=\phi(e)$ if i and j are connected by an edge e, and $a_{i,j}=0$ if i and j are not adjacent. A sign pattern matrix A is *sign-nonsingular* if any real $U\times W$ matrix with the same sign pattern as A has full column rank. Notice that swapping the signs of some rows and columns of a sign-nonsingular sign pattern matrix gives a sign-nonsingular sign pattern matrix. An example of a sign-nonsingular sign pattern matrix is $\begin{bmatrix}+ & -\\ + & +\end{bmatrix}$.

Let (G,ϕ) be a signed graph. For a subset S of V, color the vertices in S black and color the other vertices white. We now repeatedly apply the following rule. Suppose we have a set U of black vertices and a set W of white vertices, such that the only white neighbors of any vertex in U are in W. Then, if the $\{+,-,0\}$-sign pattern matrix A of (G,ϕ) on $U\times W$ is sign-nonsingular, then we color the vertices in W black. If in the end, we have colored all vertices of (G,ϕ) black, then we call S a *zero-forcing set* of (G,ϕ). The *zero-forcing number* of a signed graph (G,ϕ) is the minimum cardinality of any zero-forcing set of (G,ϕ).

The next lemma shows the invariance of the zero-forcing number under sign-equivalence. Its proof follows immediately from the fact that swapping the signs of some rows and columns of a sign-nonsingular sign pattern matrix gives again a sign-nonsingular sign pattern matrix.

If the signed graphs (G,ϕ) and (G,ψ) are sign-equivalent, then $Z(G,\phi)=Z(G,\psi)$.

The zero-forcing number of a signed graph forms an upper bound of the maximum nullity of a signed graph.

$M(G,\phi) \leq Z(G,\phi)$ for every signed graph (G,ϕ). Suppose to the contrary that $Z(G,\phi) < M(G,\phi)$. Let $A = [a_{i,j}] \in S(G,\phi)$ have nullity equal to $M(G,\phi)$. Let S be a zero-forcing set of size $Z(G,\phi)$ for (G,ϕ). Since $Z(G,\phi) < M(G,\phi)$, there exists a nonzero vector $x \in \ker(A)$ with $x_s = 0$ for each $s \in S$. Since S is a zero-forcing set, there exist sets U and W such that the sign pattern matrix B of (G,ϕ) on $U \times W$ is sign-nonsingular, and we could color the vertices in W. Since B is sign-nonsingular, the submatrix of A spanned by the rows in U and columns in W has full column rank. Hence $x_i = 0$ for each $i \in W$. Continuing this, we can conclude that $x = 0$, which is a contradiction. Hence $M(G,\phi) \leq Z(G,\phi)$.

3. The zero-forcing numbers of some signed graphs

By K_4^i we denote the signed graph (K_4,ϕ), where ϕ is a function from the edge set of K_4 into $\{-,+\}$ which assigns to exactly one edge $-$. This signed graph has exactly two positive and two negative triangles.

If in a signing (K_n,ϕ) of K_n all triangles are positive or all triangles are negative, then $M(K_n,\phi) = Z(K_n,\phi) = n-1$. For any signing (K_n,ϕ) of K_n with at least one positive and at least one negative triangle, $M(K_n,\phi) \leq Z(K_n,\phi) \leq n-2$.

Suppose first that all triangles of (K_n,ϕ) are positive. Then (K_n,ϕ) is sign-equivalent to the signed graph in which all edges are positive, so we may assume that in (K_n,ϕ) all edges are positive. It is clear that $Z(K_n,\phi) \leq n-1$. Let $A = J \in S(K_n,\phi)$, where J is the all-1 matrix. Then $M(K_n,\phi) \geq n-1$. Hence $M(K_n,\phi) = Z(K_n,\phi) = n-1$. The case where all triangles of (K_n,ϕ) are negative is similar.

Suppose now that that in the signing there is at least one negative and at least one positive triangle. We show that (K_n,ϕ) has an induced K_4^i. Choose a negative and a positive triangle that have as many vertices in common as possible. If these triangles have two vertices in common, then we are done. Suppose now that these triangles have no vertex in common. Let v_1 be a vertex in the positive triangle and v_2 a vertex in the negative triangle. Let w be a vertex in the positive triangle distinct from v_1. If the vertices w, v_1, and v_2 span a negative triangle, then we are done. If w, v_1, and v_2 span a positive triangle, then we have found a positive triangle and a negative triangle that have at least one vertex in common. Let v be the common vertex of these positive and negative triangle. Let w_1 be a vertex of the positive triangle distinct from v and let w_2 be a vertex of the negative triangle. If the triangle spanned by v, w_1, and w_2 is negative, then we are done. If the triangle spanned by v, w_1, and w_2 is positive, then we are also done.

Let v_1 and v_2 be the common vertices of a positive and negative triangle. Let w_1 and w_2 be the other vertices in these triangles. Color the vertices of $S := V(K_n) \setminus \{w_1, w_2\}$ black. Let $U = \{v_1, v_2\}$ and $W = \{w_1, w_2\}$. Then the sign pattern matrix on $U \times W$ is

$$A = [a_{i,j}] = \begin{bmatrix} - & + \\ + & + \end{bmatrix},$$

which is sign-nonsingular. Hence S is a zero-forcing set, and so $Z(K_n,\phi) \leq n-2$.

If $(K_{p,q},\phi)$ is a signing of $K_{p,q}$ in which each cycle is positive, then $M(K_{p,q},\phi) = Z(K_{p,q},\phi) = p+q-2$. We may assume that all edges of $(K_{p,q},\phi)$ are positive. Let V_1 and V_2 be the bipartition of $K_{p,q}$. We assume that $V_1 = \{1,\ldots,p\}$ and $V_2 = \{p+1,\ldots,p+q\}$. Let S be a set containing all but one vertex of V_1 and all but one vertex of V_2. Then S is a zero-forcing set for $K_{p,q}$. Hence $Z(K_{p,q},\phi) \leq Z(K_{p,q}) \leq p+q-2$.

Let J be the $p \times q$ all-1 matrix and let

$$A = [a_{i,j}] = \begin{bmatrix} 0 & J \\ J^T & 0 \end{bmatrix}.$$

Then $A \in S(K_{p,q}, \phi)$ and A has nullity $p+q-2$. Hence $M(K_{p,q}, \phi) \geq p+q-2$, and so $M(K_{p,q}, \phi) = Z(K_{p,q}, \phi) = p+q-2$.

Let $\max\{p, q\} \geq 4$. If $(K_{p,q}, \phi)$ is a signing of $K_{p,q}$ which contains a negative cycle, then $M(K_{p,q}, \phi) \leq Z(K_{p,q}, \phi) \leq p+q-4$. Let V_1 and V_2 be the bipartition of $K_{p,q}$. Let $S_1 \subseteq V_1$ with $|S_1| = p-2$ contain two vertices of C, and let $S_2 \subseteq V_2$ with $|S_2| = q-2$ contain no vertex of C. Color the vertices in $S := S_1 \cup S_2$. Then S is a zero-forcing set for $(K_{p,q}, \phi)$. Since $|S_1 \cup S_2| = p+q-4$, $Z(K_{p,q}, \phi)) \leq p+q-4$.

If a signing $(K_{3,3}, \phi)$ of $K_{3,3}$ has a negative cycle, then $M(K_{3,3}, \phi) = Z(K_{3,3}, \phi) = 3$. Let C be a negative cycle of the signing $(K_{3,3}, \phi)$. Let $V_1 = \{1, 2, 3\}$ and $V_2 = \{4, 5, 6\}$ be the bipartition of $K_{3,3}$. We may assume that the vertices of C are $1, 2, 4$, and 5. Let $U = \{1, 2\}$ and $W = \{4, 5\}$. Color the vertices in $\{1, 2, 6\}$ black. Then the white neighbors of each vertex in U belong to W. The sign pattern matrix B spanned by the rows in $\{1, 2\}$ and columns in $\{4, 5\}$ equals

$$B = [b_{i,j}] = \begin{bmatrix} + & + \\ - & + \end{bmatrix},$$

which is sign-nonsingular. Hence we can color the vertices 4 and 5 black. As a final step, we color vertex 3 black. Hence $Z(K_{3,3}, \phi) \leq 3$ for this signing of $K_{3,3}$.

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On The Edge-Balanced Index Set of $K_{13,7}$

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Abstract: Let G be a graph with vertex set V and edge set E. Consider a binary labeling of E, called an edge-friendly labeling, where the absolute difference in the number of edges labeled 1 and 0 is no more than one. When the degree of every vertex is odd, an edge-friendly labeling induces a binary labeling on the vertices as follows: a vertex that is incident with more 1-edges than 0-edges is labeled 1; otherwise, the vertex is labeled 0. The edge-balanced index set of G is the collection of the absolute differences in the number 1-vertices and the number of 0-vertices over all edge-friendly labellings of G. In this paper, we determine the edge-balanced index set of the complete bipartite graph $K_{13,7}$ as a next step toward a general solution to a problem introduced by Kong, Wang, and Lee and later examined by Krop and Sikes.

1. Introduction

1.1. **Definitions.** In [2], Gallian provides an introduction to graph labellings, including definitions and results. More on graph theory in general can be found in Diestel's graduate text [1]. A *complete bipartite graph*, denoted $K_{m,n}$, is a graph whose vertex set V can be partitioned into two nonempty subsets A and B, where $|A| = m$ and $|B| = n$. We assume A and B are chosen so that $m \geq n$. Every vertex in part A is adjacent to every vertex in part B, but there are no adjacencies within a part. In other words, if E denotes the edge set of $K_{m,n}$, then $E = \{ab : \forall a \in A, \forall b \in B\}$ and $|E| = mn$. A binary *edge labeling* is a labeling of every edge by either 1 or 0. Edges labeled 1 are called 1-*edges* and those labeled 0 are called 0-*edges*. For such an edge labeling, let $e(1)$ be the number of 1-edges and let $e(0)$ be the number of 0-edges, and note that $e(1) + e(0) = |E|$. If $|e(1) - e(0)| \leq 1$, we have an *edge-friendly labeling*, and the graph is said to be *edge-friendly*.

The edge labeling induces a partial *vertex labeling* as follows: for a particular vertex v, if the incident edges are such that the number of 1-edges, denoted $\deg_1(v)$, is greater than the number of 0-edges, denoted $\deg_0(v)$, then the vertex label is 1; if $\deg_0(v) > \deg_1(v)$, then the vertex label is 0; otherwise, the vertex is unlabeled. We call a vertex labeled 1 a 1-*vertex* and a vertex labeled 0 a 0-*vertex*. Let $v(1)$ be the number of 1-vertices and $v(0)$ be the number of 0-vertices. The *edge-balanced index set* for a graph G is defined as

$$EBI(G) = \{|v(0) - v(1)| : \text{over all edge-friendly labellings of } G\}.$$

1.2. **History and Motivation.** One of the first questions we asked was: What is the "smallest" graph $K_{m,n}$, where m and n are both positive odd integers, for which $EBI(K_{m,n})$ is unknown?

Kong, Wang and Lee introduced the problem of finding $EBI(K_{m,n})$ in [3] by solving the cases where $n = 1, 2, 3, 4, 5$ and $m = n$. In [4], Krop and Sikes generalized some of these results by tackling the family of complete odd bipartite graphs where m and n do not differ by too much, i.e., when n is more than about $\frac{1}{2}m$. In fact, Krop and Sikes prove that for

odd $m > 5$,

$$EBI(K_{m,m-2a}) = \{0, 2, \ldots, 2m - 2a - 8, 2m - 2a - 6\},$$

where $1 \leq a \leq \frac{m-3}{4}$.

Then the "smallest" complete odd bipartite graph for which EBI is unknown is $K_{13,7}$, and that is where we begin our study.

2. Finding the Edge-Balanced Index Set of $K_{13,7}$

An observation from [3] is helpful in determining elements in the edge-balanced index set for graphs in which all of the vertices have odd degree: If G is a graph whose vertices all have odd degree, then $EBI(G)$ contains only even integers. Since all vertices of $K_{13,7}$ have degree 13 or degree 7, we are assured that $EBI(K_{13,7})$ contains only even integers.

$EBI(K_{13,7}) = \{14, 12, 10, 8, 6, 4, 2, 0\}$.

We prove this theorem in three stages. In the first part of the proof, Section 2.1, we show that 20, 18, and 16 are not elements of the edge-balanced index set of $K_{13,7}$. In the second part, Section 2.2, we show that $14 \in EBI(K_{13,7})$ and exhibit an edge-friendly labeling that achieves this value. In the final part of the proof, Section 2.3, we provide additional edge-friendly labellings that give each of the smaller indices in $EBI(K_{13,7})$.

2.1. **Showing $20, 18, 16 \notin EBI(K_{13,7})$.** Consider any edge-friendly labeling of $K_{13,7}$. Then $e(1) + e(0) = 13 \cdot 7 = 91$. The degree of every vertex of $K_{13,7}$ is odd and every edge is labeled, so every vertex is labeled either 1 or 0, implying $v(1) + v(0) = 20$. Without loss of generality, we assume that $v(1) \geq v(0)$. We claim that $v(1) - v(0)$ cannot be equal to 20, 18, or 16. Since $v(1) + v(0) = 20$, we find that $v(1) - v(0) = 20$ if and only if $v(1) = 20$ and $v(0) = 0$. Likewise, $v(1) - v(0) = 18$ if and only if $v(1) = 19$ and $v(0) = 1$ and $v(1) - v(0) = 16$ if and only if $v(1) = 18$ and $v(0) = 2$. Thus, to prove the claim we show that there is no edge-friendly labeling of $K_{13,7}$ for which $v(0)$ is 0, 1, or 2.

To obtain an edge-friendly labeling of $K_{13,7}$, there must be at least two 0-vertices in part A and at least one 0-vertex in part B. Proving this is equivalent to showing that $v(0) \geq 3$; that is, $v(0) \neq 0, 1, 2$. First, assume there exists an edge-friendly labeling of $K_{13,7}$ such that there are no 0-vertices in A. Then for every vertex $v \in A$, we have that $\deg_1(v) \geq 4$. Counting the 1-edges incident with the vertices in A, we find that $e(1) \geq 13 \cdot 4 = 52$ and $e(0) \leq 91 - 52 = 39$, so that $e(1) - e(0) \geq 13$, which contradicts the edge-friendliness assumption. Similarly, if we assume there exists an edge-friendly labeling of $K_{13,7}$ such that there is exactly one 0-vertex in A, then $e(1) \geq 12 \cdot 4 = 48$, $e(0) \leq 91 - 48 = 43$, and $e(1) - e(0) \geq 5$, a contradiction. If there exists an edge-friendly labeling of $K_{13,7}$ such that there are no 0-vertices in B, then for every vertex $v \in B$, $\deg_1(v) \geq 7$. Counting the 1-edges incident with the vertices in B reveals that $e(1) \geq 7 \cdot 7 = 49$ and $e(0) \leq 91 - 49 = 42$, implying $e(1) - e(0) \geq 7$, another contradiction. Since there must be at least two 0-vertices in part A and at least one in part B, we have that $v(0) \neq 0, 1, 2$.

Having shown that 20, 18, and 16 are not elements of $EBI(K_{13,7})$, we provide next an edge-friendly labeling that gives $v(1) - v(0) = 14$, which is the maximal element in $EBI(K_{13,7})$.

2.2. **The $\max\{EBI(K_{13,7})\}$ labeling.** For the remainder of the proof, we name the vertices in part A as $\{u_1, \ldots, u_{13}\}$ and those in part B as $\{v_1, \ldots, v_7\}$. Label the following edges by 1:

- (v_2, u_i) for $i = 3, 4, \ldots, 9$
- (v_3, u_i) for $i = 10, 11, 12, 13, 3, 4, 5$
- (v_4, u_i) for $i = 6, 7, \ldots, 12$

- (v_5, u_i) for $i = 13, 3, 4, \ldots, 8$
- (v_6, u_i) for $i = 9, 10, \ldots, 13, 3, 4$
- (v_7, u_i) for $i = 5, 6, \ldots, 11$
- (v_1, u_i) for $i = 12, 13, 3, 4$

Label all remaining edges of $K_{13,7}$ by 0. See Figure 1 and note that only the 1-edges are shown.

[xscale=0.95,yscale=0.8] vertex1=[circle,fill=black,inner sep=3pt]
vertex2=[circle,fill=black!40 ,inner sep=3pt]
in 1,2 [vertex2,label=above:u] (u-) at (,4) ;
in 3,...,13 [vertex1,label=above:u] (u-) at (,4) ;
in 1 [vertex2,label=below:v] (v-) at (,0) ;
in 2,...,7 [vertex1,label=below:v] (v-) at (,0) ;
in 3,4,5,6,7,8,9 (v-2) – (u-);
in 10,11,12,13,3,4,5 (v-3) – (u-);
in 6,7,8,9,10,11,12 (v-4) – (u-);
in 13,3,4,5,6,7,8 (v-5) – (u-);
in 9,10,11,12,13,3,4 (v-6) – (u-);
in 5,6,7,8,9,10,11 (v-7) – (u-);
in 12,13,3,4 (v-1) – (u-);

Figure 1. An edge-friendly labeling of $K_{13,7}$ that shows $14 \in EBI(K_{13,7})$. The black vertices represent 1-vertices and the gray vertices represent 0-vertices.

Counting the 1-edges and 0-edges incident with the vertices in part B gives $e(1) = 6 \cdot 7 + 4 = 46$ and $e(0) = 91 - 46 = 45$, so that $K_{13,7}$ is edge-friendly. By construction, $\deg_1(u_i) \geq 4$ for $3 \leq i \leq 13$ and $\deg_1(v_i) = 7$ for $2 \leq i \leq 7$, which means that this edge-friendly labeling induces a vertex labeling where $\{v_3, v_4, \ldots, v_{13}\}$ and $\{u_2, u_3, \ldots, u_7\}$ are 1-vertices. Moreover, $\deg_0(u_1) = \deg_0(u_2) = 7$ and $\deg_0(v_1) = 9$, which means that $\{v_1, v_2, u_1\}$ are 0-vertices. Thus, $v(1) - v(0) = 17 - 3 = 14 = \max\{EBI(K_{13,7})\}$.

Having shown that $\max\{EBI(K_{13,7})\} = 14$ and given an edge-friendly labeling that achieves this index, we conclude the proof of Theorem 2 by providing edge-friendly labellings that yield each of the smaller elements in $EBI(K_{13,7})$.

2.3. **Smaller indices.** To show that the smaller indices, $12, 10, \ldots, 0$, are in $EBI(K_{13,7})$, we reallocate edge labels by exchanging pairs of labels on 1-edges and 0-edges. Note that we are not adding or deleting any 1-edges or 0-edges, but rather switching the labels on certain edges so as to maintain the edge-friendliness provided by the $\max\{EBI(K_{13,7})\}$ labeling. We perform the switches successively using this labeling as the starting point.

First, switch the labels on 0-edge (u_1, v_2) and 1-edge (v_2, u_5). This switch does not change the labels on vertices u_1 and v_2 but changes u_5 from a 1-vertex to a 0-vertex. We now have $v(1) = 16$, $v(0) = 4$, and $12 \in EBI(K_{13,7})$. Second, switch the labels on 0-edge (u_1, v_3) and 1-edge (v_3, u_{10}). The labels on vertices u_1 and v_3 do not change but u_{10} changes from a 1-vertex to a 0-vertex. Thus, $v(1) = 15$, $v(0) = 5$, and $10 \in EBI(K_{13,7})$. Continuing in this manner we perform switches of the labels on the following edges:

- (u_1, v_4) and (v_4, u_6), giving $8 \in EBI(K_{13,7})$
- (u_2, v_5) and (v_5, u_8), giving $6 \in EBI(K_{13,7})$
- (u_2, v_6) and (v_6, u_9), giving $4 \in EBI(K_{13,7})$
- (u_2, v_7) and (v_7, u_7), giving $2 \in EBI(K_{13,7})$

Finally, we switch 0-edge (v_1, u_1) and 1-edge (u_1, v_2) to obtain $0 \in EBI(K_{13,7})$.

This concludes the proof of Theorem 2.

3. Future Work

Currently, we are trying to generalize the results presented for $K_{13,7}$ to solve the problem of finding $EBI(K_{m,n})$ for all positive odd integers $m \geq n$. Additional future work will include examining the cases where m is even and/or n is even.

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Calculating Gröbner Bases in a Polynomial Ring

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Abstract The concept of **Grö bner basis** is relatively new. Though Bruno Buchberger first introduced the concept of Grö bner basis in his doctoral thesis in 1965, the concept remained unknown to the research community until a decade later. The theory of Grö bner basis has become an important subarea in computer algebra. It is included in all the major program systems of symbolic computation, and it is being fruitfully applied to a variety of research areas. Having developed a basic understanding of what a Grö bner basis is, I calculated Grö bner bases over given polynomial rings using the Multivariate Division Algorithm.

1. Introduction

The growth in the capacity of computers since the 1960s has certainly contributed to an increase in interest in constructiveness and concrete calculations. The area of algorithms and computer softwares are examples of such major developments.

The work of Bruno Buchberger on Grö bner bases has been generating increasing interest because of its usefulness in providing computational tools that can be used to study problems in mathematics, science, engineering, and computer science.

For the purpose of this work, I review(in the Preliminary section of this paper)several relevant concepts from the theory of rings and fields among other relevant resources, and discuss Hilbert's Basis Theorem. At the center of Grö bner bases theory is the multivariate division algorithm, an analogue of the univariate division algorithm. Just like the division algorithm in the univariate case, the multivariate division algorithm can be used to produce sets of generators for ideals in the ring of multivariate polynomials with some properties. That is, Grö bner bases are the analog of gcd in the multivariate case in that if $G = \{g_1, ..., g_t\}$ is a Grö bner basis for an ideal $I \subseteq k[x_1, ..., x_n]$, then
$f \in k[x_1, ..., x_n]$ is in I if and only if f is reduced to 0 modulo G.

I conclude this work with some selected examples showing how to calculate Grö bner bases in polynomial rings.

2. Preliminaries

Definition 1. A **ring** $\langle R, +, . \rangle$ is a non-void set R together with two binary operations,"+" and "$\cdot$" of addition and multiplication defined on R such that the following axioms are satisfied:

(1) $\langle R, +\rangle$is an Abelian group.
(2) Multiplication is associative on R.

(3) For all $a, b, c \in R$, the
left distributive law, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, and the
right distributive law, $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$,
hold.

Definition 2. If there is an element 1 in the ring R such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$, then R is called a **ring with unity**.

Definition 3. The ring R is said to be **commutative** if and only if for all $a, b \in R$, $a \cdot b = b \cdot a$.

Definition 4. An element $r \neq 0$ in a ring R is called a **zerodivisor** in R if $rs = 0$ for some element $s \in R$, $s \neq 0$. An element $r \neq 0$ which is not a zerodivisor is called a **nonzerodivisor**.

Definition 5. An **integral domain** is a zero-divisor-free commutative ring with unity.

Definition 6. A **field** is a commutative ring with unity in which every element $a \neq 0$ has a multiplicative inverse a^{-1}, so that
$aa^{-1} = a^{-1}a = 1$.

Theorem 1: Every field is an integral domain.
Proof:
Let A be a field. Let $a, b \in A$ such that $ab = 0$. Consider $b \neq 0$. Then $a = a \cdot 1 = a(bb^{-1}) = (ab)b^{-1} = 0 \cdot b^{-1} = 0$.

Theorem 2: Every integral domain with a finite number of elements is a field.
Proof:
Let $R = \{x_1, x_2, ..., x_n\}$ be a finite integral domain and take $x_i \neq 0$. All elements of R can be written as $\{x_i x_k \mid k = 1, ..., n\}$. We need to show that $x_i x_k \neq x_i x_l$ if $k \neq l$. This is true, for if $x_i x_k = x_i x_l$ then $x_i(x_k - x_l) = 0$ which gives $x_k - x_l = 0$ since $x_i \neq 0$, so $x_k = x_l$. Since every element in R can be written as $x_i x_k$ for some k, we get that $x_i x_k = 1$ for some k so x_i has a multiplicative inverse.

Definition 7. The **characteristic** of a ring R, denoted char (R), is the smallest positive integer n such that $n \cdot 1 = 0$. If no such integer n exists, then the characteristic of the ring is 0.

Lemma 1: Let k be a ring and u be an indeterminate. Then $k[u]$, the set of all polynomials in u with coefficients in k, is a ring under the usual polynomial addition and multiplication.

Proof (Outline):

It is clear that $\langle k[u], + \rangle$ is an Abelian group. The associative and distributive laws are straightforward, but require slightly tedious calculations. We will show this by proving the associative law.

Let $a_i, b_j, c_l \in k$. Then by the ring axioms, we have:

$$\left[\left(\sum_{i=0}^{\infty} a_i u^i\right)\left(\sum_{j=0}^{\infty} b_j u^j\right)\right]\left(\sum_{l=0}^{\infty} c_l u^l\right) = \left[\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)u^n\right]\left(\sum_{l=0}^{\infty} c_i u^l\right)$$
$$= \sum_{s=0}^{\infty}\left[\sum_{n=0}^{s}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)c_{s-n}\right]u^s$$
$$= \sum_{s=0}^{\infty}\left(\sum_{s=i+j+l} a_i b_j c_l\right)u^s$$
$$= \sum_{s=0}^{\infty}\left[\sum_{m=0}^{s} a_{s-m}\left(\sum_{j=0}^{m} b_j c_{m-j}\right)\right]u^s$$
$$= \left(\sum_{i=0}^{\infty} a_i u^i\right)\left[\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} b_j c_{m-j}\right)u^m\right]$$
$$= \left(\sum_{i=0}^{\infty} a_i u^i\right)\left[\left(\sum_{j=0}^{\infty} b_j u^j\right)\left(\sum_{l=0}^{\infty} c_l u^l\right)\right].$$

Similarly the distributive law can be verified.

Note: If k is commutative, then so is $k[u]$. To see this, observe that if $f, g \in k[u]$ with $f(x) = a_0 + a_1 x + ... + a_n x^n + ...$, and
$g(x) = b_0 + b_1 x + ... + b_n x^n + ...$, then $(fg)(x) = d_0 + d_1 x + ... + d_n x^n + ...$,
where $d_n = \sum_{i=0}^{n} a_i b_{n-i}$.
It is clear that except for a finite number of values of i, the d_i are zeros. So the definition makes sense, and unless k is commutative,
$\sum_{i=0}^{n} a_i b_{n-i}$ may not be equal to $\sum_{i=0}^{n} b_i a_{n-i}$.

Theorem 3: If k' is a ring, $k \subseteq k'$ a subring and $u \subseteq k$ a subset, then $k[u]$ is the subring of k' generated by k and u. In fact, if v is another subset of k, then $k[u][v] = k[u \cup v]$ is a subring of k'.

Proof:
First, it is clear that $k[u \cup v] \supseteq k[u], v$. Since the subring generated by $k[u]$ and v is contained in every subring containing these sets, we have $k[u \cup v] \supseteq k[u][v]$. Next, it is clear that $k[u][v]$ contains k and the subset $u \cup v$. Hence, $k[u][v] \supseteq k[u \cup v]$.
Therefore, $k[u][v] = k[u \cup v]$.

Inductively from the above result it follows that
$k[u_1, u_2, ..., u_n] = k[u_1][u_2]...[u_n]$. That is, $k[u_1, u_2, ..., u_n]$ is obtained from k by a succession of adjunctions of single elements to previously constructed subrings.

Note:

(1) A polynomial f in $x_1, ..., x_n$ with coefficient in k is a finite linear combination of monomials.
We write: $f = \sum_{\alpha=0}^{n} a_\alpha x^\alpha, a_\alpha \in k$, where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$.
$k[x_1, ..., x_n]$:= the set of all polynomials in $x_1, ..., x_n$ with coefficients in k.

(2) $k[x_1, ..., x_n]$ is not a field; it is rather a commutative ring.
e.g. $f(x_1) = x_1 \Rightarrow f \in k[x_1, ..., x_n]$ but $\frac{1}{x_1} \notin k[x_1, ..., x_n]$.

Definition 8. A nonvoid subset I of $k[x_1, ..., x_n]$, is an **ideal** in $k[x_1, ..., x_n]$ if and only if:

(1) $f, g \in I$ implies that $f + g \in I$.
(2) $f \in I$ and $h \in k[x_1, ..., x_n]$ implies that $hf \in I$.

Identifying all the generators of an ideal is very important in polynomial ideal theory and Hilbert Basis Theorem (stated below), one of the important results in polynomial ideal theory, states that any ideal in $k[x_1, ..., x_n]$ is finitely generated.

Theorem 4 (Hilbert Basis Theorem): Every ideal $I \subset k[x_1, ..., x_n]$ is finitely generated.

Before we prove the Hilbert Basis Theorem we need to state and prove the following lemmas.

Lemma 2: For a ring k, the following conditions are equivalent.

(1) If $I \subseteq k$ is an ideal, then there exists functions $f_1, f_2, ..., f_t \in k$ such that $I = \langle f_1, ..., f_t \rangle$.

(2) If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an ascending chain of ideals in k, then there exists an N such that $I_N = I_{N+1} = \cdots$.

Proof:
($\Leftarrow$) Assume (2). Suppose to the contrary that for some ideal $I \subseteq k$, I is not generated by a finite set of elements in k. Let $f_1 \in I$. Then there exists $f_2 \in I$ with $f_2 \notin \langle f_1 \rangle$. So $\langle f_1 \rangle \subset \langle f_1, f_2 \rangle$. Similarly there exists $f_3 \in I$ with $f_3 \notin \langle f_1, f_2 \rangle$. So $\langle f_1, f_2 \rangle \subset \langle f_1, f_2, f_3 \rangle$. Now continuing this process we get a strictly ascending chain of ideals of k which contradicts (2). Thus (1) holds.
($\Rightarrow$) Now, assume (1), and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an ascending chain of ideals of k. Consider the set $I = \bigcup_{i=1}^{\infty} I_i$. Since the ideals are ascending, it follows that I is also an ideal of k. So condition (1) $\Rightarrow I = \langle f_1, f_2, ..., f_t \rangle$ for some $f_1, f_2, ..., f_t \in k$. Since $f_i \in I$, for $i = 1, 2, ..., t$, there exists N_i such that $f_i \in I_{N_i}$. Let $N = \underset{1 \le i \le t}{max}\{N_i\}$. Then $f_i \in I_N$ for all $i = 1, 2, ..., t$ and so $I \subseteq I_N$. Thus, $I = I_N$ and condition (2) holds.

Note: Condition 2 in Lemma 2 is called the ascending chain condition.

Lemma 3: If a commutative ring k has the ascending chain condition, then so does $k[x]$.

Proof:
Let k be a commutative ring and x be an indeterminate. So the note after Lemma 1 implies that $k[x]$ is also a commutative ring. Suppose k has the ascending chain condition. Let $I \subseteq k[x]\backslash\{\emptyset\}$ be an arbitrarily fixed ideal. By Lemma 2, it is sufficient to show that I is finitely generated. For each $n \geq 0$, define

$$I_n := \{a \in k \mid a \text{ is a coefficient of } x^n, ax^n + \cdots + a_0 \in I\} \cup \{0\}.$$

Observe that I_n is an ideal of k and that $I_n \subseteq I_{n+1}$ for all $n \geq 0$. Since k has the ascending chain condition, there exists N such that $I_n = I_N$ for all $n \geq N$. By Lemma 2, each $I_i = \langle a_{i_1}, a_{i_2}, ..., a_{i_{s_i}} \rangle$ is finitely generated where $i = 1, 2, ..., N$ and $j = 1, 2, ..., s_i$. Let $f_{i_j} \in I$ be a polynomial of degree i with leading coefficients a_{i_j}, i.e. $a_{i_j} = lc(f_{i_j})$. To complete the proof, we need to show that $I = \langle f_{i_j} | 1 \leq i \leq N, 1 \leq j \leq s_i \rangle$.
Let $I^* = \langle f_{i_j} | 1 \leq i \leq N, 1 \leq j \leq s_i \rangle$. Then since f_{i_j} is a polynomial of degree i in I with coefficients a_{i_j}, it follows that $I^* \subseteq I$.
Next, let $f \in I$ be arbitrarily fixed and suppose that the degree of f is n. We must show by the principle of mathematical induction that $f \in I^*$.
If $n = 0$ or f is the zero polynomial then trivially $f \in I_0$ and thus, $f \in I^*$. Now let $n > 0$, and assume that all the polynomials in I of degree at most $n - 1$ are also in I^*.
Let $a = lc(f)$. If $n \leq N$, then for some $t_j \in k$, $a \in I_n$, which implies that $a = \sum_{j=1}^{s_n} t_j a_{n_j}$. So the polynomial $g = \sum_{j=1}^{s_n} t_j f_{n_j}$ of degree n, with $lc(g) = a$ is in I^*. So the $deg(f - g) \leq n - 1$ and $f - g \in I$.
By the Principle of Mathematical Induction, $f - g \in I^*$ and so $f \in I^*$ as well. Finally, if $n > N$, then $a \in I_n = I_N$, and $a = \sum_{j=1}^{s_N} t_j a_{N_j}$, for some $t_j \in k$. The polynomial $g = \sum_{j=1}^{s_N} t_j x^{n-N} f_{N_j}$ with leading coefficients a and having degree n is in I^*. Hence, the $deg(f - g) \leq n - 1$. Thus, by induction, $f - g \in I^*$. Therefore, $f \in I^*$.

Proof (Hilbert Basis Theorem):
By simple induction and Lemma 2 we see that $k[x_1, ..., x_n]$ has the ascending chain condition, since k has the ascending chain condition. Hence, the Hilbert Basis Theorem holds by Lemma 1, Lemma 3, and the note following Lemma 1.

3. Multivariate Division Algorithm

To create a division algorithm for multivariate polynomials, we need to define concepts similar to those of degree of a polynomial and leading term.

A **total order** $<$ on a set S is an order such that for every $j, k \in S$ exactly one of the following relations must hold:
$j < k$, $j = k$, or $k < j$.
Let $\mathbb{N}^n = \{(\alpha_1, ..., \alpha_n) | \alpha_i \in \mathbb{N}, i = 1, ..., n\}$ and denote the set of all power products as $\mathbb{T}^n$. For the next definition, denote $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by x^α, where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$.

Definition 9. A **term order** on $\mathbb{T}^n$ is a total order $<$ on $\mathbb{T}^n$ such that

(1) $1 < x^\alpha$ for all $x^\alpha \in \mathbb{T}^n$, $x^\alpha \neq 1$.
(2) If $x^\alpha < x^\gamma$, then $x^\alpha x^\beta < x^\gamma x^\beta$, for all $x^\beta \in \mathbb{T}^n$.

Note that unless an order on the set of variables $\{x_1, ..., x_n\}$ has been given, a term order is meaningless . Unless otherwise noted, we will assume that $x_1 > x_2 > \cdots > x_n$. This merely means that x_1 is our largest or first variable, x_2 the next largest variable, and so on.

The following is a definition of a term order that is commonly used.

Definition 10. Lexicographical ordering (denoted, "$>_{lex}$"). For $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n) \in I$ we write $x^\alpha >_{lex} x^\beta$ if and only if the first coordinates α_i and β_i in α and β from the left (which are distinct) satisfy $\alpha_i >_{lex} \beta_i$ (i.e. α_i precedes β_i lexicographically).

For all $f \in k[x_1, ..., x_n]$, f can be written as $f = u_1 x_1^\alpha + u_2 x_2^\alpha + \cdots + u_r x_r^\alpha$ where $\alpha_i \in k \backslash \{0\}$, x^{α_i} are power products, and
$x_1^\alpha >_{lex} x_2^\alpha >_{lex} \cdots >_{lex} x_r^\alpha$.

Definition 11. The **leading power product** of f is denoted
$lp(f) = x_1^\alpha$.

Definition 12. The **leading coefficient** of f is denoted $lc(f) = u_1$.

Definition 13. The **leading term** of f is denoted lt(f) $= u_1 x_1^\alpha$.

The multivariate division algorithm consists of a sequence of reduction steps.

Definition 14. Let $f, g, h \in k[x_1, ..., x_n]$ with $g \neq 0$. We say that f **reduces** to h modulo g in one step, denoted

$$f \xrightarrow{g} h,$$

if and only if $lp(g)$ divides a non-zero term ax^α that appears in f and

$$h = f - \frac{ax^\alpha}{lt(g)} g.$$

In the multivariate case, it may be the case that we have to divide by more than one polynomial at a time. We extend the previous definition to include this possibility.

Definition 15. Let f, h, and $f_1, ..., f_s$ be polynomials in $k[x_1, ..., x_n]$ with $f_i \neq 0$ for $i = 1, ..., s$. Let $F = \{f_1, ..., f_s\}$. We say that f **reduces** to h modulo F, denoted

$$f \xrightarrow{F}_+ h,$$

if and only if there exists a sequence of indices $i_1, i_2..., i_t \in \{1, ..., s\}$ and a sequence of polynomials $h_1, ..., h_{t-1} \in k[x_1, ..., x_n]$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h.$$

Definition 16. A polynomial r is called **reduced** with respect to a set of non-zero polynomials $F = \{f_1, ..., f_s\}$ if $r = 0$ or no power product that appears in r is divisible by any one of the $lp(f_i)$, $i = 1, ..., s$. In other words, r cannot be reduced modulo F.

Definition 17. If $f \xrightarrow{F}_+ r$ and r is reduced with respect to F, then we call r a **remainder** for f with respect to F.

Multivariate Division Algorithm

INPUT: $f, f_1, ..., f_s \in k[x_1, ..., x_n]$ with $f_i \neq 0$ for all i.

OUTPUT: $\alpha_1, ..., \alpha_s, r$ such that $f = \sum_{i=1}^{s} a_i f_i + r$ and r is reduced with respect to $\{f_1, ..., f_s\}$ and $\underset{1 \leq i \leq s}{max}\{lp(\alpha_i)lp(f_i), lp(r)\} = lp(f)$

INITIALIZATION: $\alpha_1 := 0$, $\alpha_2 := 0$, ..., $\alpha_s := 0$, $r := 0$, $h := f$

WHILE $h \neq 0$ DO

IF there exists i such that $lp(f_i)$ divides $lp(h)$ THEN

Choose i least such that $lp(f_i)$ divides $lp(h)$

$\alpha_i := \alpha_i + \frac{lt(h)}{lt(f_i)}$

$h := h - \frac{lt(h)}{lt(f_i)} f_i$

ELSE

$r := r + lt(h)$

$h := h - lt(h)$

4. Gröbner Bases and Examples

Definition 18. Let $I \subseteq k[x_1, ..., x_n]$ be an ideal and $G = \{g_1, ..., g_t\} \subseteq I$ be a set of non-zero polynomials. G is called a **Gröbner basis** for I if and only if for each $f \in I \backslash \{0\}$, there exists $i \in \{1, ..., t\}$ such that $lp(g_i)$ divides $lp(f)$.

Definition 19. For a subset S of $k[x_1, ..., x_n]$, the **leading term ideal** of S is the ideal $Lt(S) = \langle lt(s) | s \in S \rangle$.

Theorem 5: Let $G = \{g_1, ..., g_t\}$ be a set of non-zero polynomials in $k[x_1, ..., x_n]$ and $I = \langle G \rangle$. Then the following statements are equivalent.

(1) G is a Gröbner basis for I.

(2) $f \in I$ if and only if $f \xrightarrow{G}_+ 0$.

(3) $f \in I$ if and only if $f = \sum_{i=1}^{t} h_i g_i$ with $lp(f) = max(lp(h_i)lp(g_i))$.

(4) $Lt(G) = Lt(I)$.

(5) for all $f \in k[x_1, ..., x_n]$, if $f \xrightarrow{G}_+ r_1$ and $f \xrightarrow{G}_+ r_2$ and r_1, r_2 are reduced with respect to G, then $r_1 = r_2$.

Proof:
See [3].

Thus, we have the important outcome that a Gröbner basis
$G = \{g_1, ..., g_t\}$ for I is a set of generators for I. That is, $I = \langle g_1, ..., g_t \rangle$.

Another important result is the fact that every nonzero ideal
$I \subseteq k[x_1, ..., x_n]$ has a Gröbner basis.
Given a set of generators $f_1, ..., f_s$ of an ideal $I \subseteq k[x_1, ..., x_n]$, Buchberger's Algorithm produces a Gröbner basis for I. We recall that such a finite set of generators for I always exists by Hilbert's Basis Theorem.

Proposition 1. Let $G = \{g_1, ..., g_t\}$ be a Gröbner basis for an ideal $I \subset k[x_1, ..., x_n]$ and let $f \in k[x_1, ..., x_n]$. Then there is a unique
$r \in k[x_1, ..., x_n]$ with the following two properties:

(1) No term of r is divisible by any of $lt(g_1), ..., lt(g_t)$.
(2) There exists a $g \in I$ such that $f = g + r$.

In particular, r is the remainder of f divided by G no matter how the elements of G are listed when using the division algorithms.

Proof:
See [3].

Definition 20. Let $L = lcm(lp(f), lp(g))$. The **S-polynomial** of f and g is defined to be

$$S(f, g) = \frac{L}{lt(f)} f - \frac{L}{lt(g)} g.$$

Even when the term order is fixed, Gröbner bases are not guaranteed to be unique.

Theorem 6: Let G be a Gröbner basis for $I \subseteq k[x_1, ..., x_n]$ and $p \in G$ be a polynomial such that $lt(p) \in \langle lt(G \backslash \{p\}) \rangle$. Then $G \backslash \{p\}$ is also a Gröbner basis for I.
Proof:
We know that $\langle lt(G) \rangle = \langle lt(I) \rangle$. Now if $lt(p) \in \langle lt(G \backslash \{p\} \rangle$, then we have $\langle lt(G \backslash \{p\} \rangle = \langle lt(G) \rangle$. By definition, it follows that $G \backslash \{p\}$ is also a Gröbner basis for I.

For uniqueness to be achieved, the following definition is employed.

Definition 21. Let $G = \{g_1, ..., g_t\}$ be a Gröbner basis. We say that G is **reduced** if and only if for all i, no non-zero term in g_i is divisible by any $lp(g_j)$ for any $j \neq i$.

Now we are ready to introduce Buchberger's Algorithm.

Buchberger's Algorithm

INPUT: $F = \{f_1, ..., f_s\} \subseteq k[x_1, ..., x_n]$ with $f_i \neq 0$ for all i.
OUTPUT: $G = \{g_1, ..., g_t\}$, a Gröbner basis for $I = \langle f_1, ..., f_s \rangle$
INITIALIZATION: $G := F$, $\mathcal{G} := \{\{f_i, f_j\} | f_i \neq f_j \in G\}$
WHILE: $\mathcal{G} \neq \emptyset$ DO
Choose any $\{f, g\} \in \mathcal{G}$.
$\mathcal{G} := \mathcal{G} - \{\{f, g\}\}$
$S(f, g) \xrightarrow{G}_+ h$, where h is reduced with respect to G.
IF $h \neq 0$ THEN
$\mathcal{G} := \mathcal{G} \cup \{\{u, h\} |$ for all u $\in G\}$
$G := G \cup \{h\}$

In practice, it is convenient to use a computer to calculate Gröbner bases. However, since the following three examples that I have created utilize functions in three variables, I chose to calculate the Gröbner bases of the associated ideals by hand.

Example 1

Let $f_1 = x + y + z$, $f_2 = xy^2z \in \mathbb{Q}[x, y, z]$ and $x >_{lex} y >_{lex} z$, be the term order.

INITIALIZATION: $G := \{f_1, f_2\}$, $\mathcal{G} = \{\{f_1, f_2\}\}$

Step 1.
Choose $\{f_1, f_2\}$.
$\mathcal{G} := \emptyset$
$S(f_1, f_2) = \frac{xy^2z}{x}(x + y + z) - \frac{xy^2z}{xy^2z}(xy^2z) = y^3z + y^2z^2 = h$, which is reduced with respect to G since neither $lp(f_1)$ nor $lp(f_2)$ divides h.
Since $h \neq 0$, let $f_3 := y^3z + y^2z^2$.
$\mathcal{G} := \{\{f_1, f_3\}, \{f_2, f_3\}\}$
$G := \{f_1, f_2, f_3\}$

Step 2.
Choose $\{f_1, f_3\}$.
$\mathcal{G} := \{f_2, f_3\}$
$S(f_1, f_3) = \frac{xy^3z}{z}(x + y + z) - \frac{xy^3z}{y^3z}(y^3z + y^2z^2) = y^4z + y^3z^2 - xy^2z^2 = -y^2z^2(x + y + z) + (z + y)(y^3z + y^2z^2) = -y^2z^2f_1 + (z + y)f_3$.
Therefore, since $S(f_1, f_3) = -y^2zf_1 + (z + y)f_3, S(f_1, f_3) \xrightarrow{G}_+ 0 = h$.

Step 3.
Choose $\{f_2, f_3\}$.
$\mathcal{G} := \emptyset$
$S(f_2, f_3) = \frac{xy^3z}{xy^2z}(xy^2z) - \frac{xy^3z}{y^3z}(y^3z + y^2z^2) = -xy^2z^2 = -z(xy^2z) = -zf_2$.
Therefore, since $S(f_2, f_3) = -zf_2, S(f_2, f_3) \xrightarrow{G}_+ 0 = h$.

Thus, the algorithm ends. $G = \{f_1, f_2, f_3\}$ is our desired Gröbner basis.

Example 2

Let $f_1 = x + y + z$, $f_2 = xy^2z \in \mathbb{Q}[x, y, z]$ and $z >_{lex} y >_{lex} x$, be the term order.

INITIALIZATION: $G := \{f_1, f_2\}$, $\mathcal{G} = \{\{f_1, f_2\}\}$

Step 1.
Choose $\{f_1, f_2\}$.
$\mathcal{G} := \emptyset$
$S(f_2, f_1) = \frac{xy^2z}{xy^2z}(xy^2z) - \frac{xy^2z}{z}(x + y + z) = -x^2y^2 - xy^3 = h$, which is reduced with respect to G since neither $lp(f_1)$ nor $lp(f_2)$ divides h.
Since $h \neq 0$, let $f_3 := -x^2y^2 - xy^3$.
$\mathcal{G} := \{\{f_1, f_3\}, \{f_2, f_3\}\}$
$G := \{f_1, f_2, f_3\}$

Step 2.
Choose $\{f_1, f_3\}$.
$\mathcal{G} := \{f_2, f_3\}$
$S(f_1, f_3) = \frac{xy^3z}{z}(x+y+z) - \frac{xy^3z}{-xy^3}(-x^2y^2 - xy^3) = x^2y^3 + xy^4 - x^2y^2z = -x^2y^2(x+y+z) + (-x-y)(-x^2y^2 - xy^3) = -x^2y^2f_1 + (-x-y)f_3$.
Therefore, since $S(f_1, f_3) = -x^2y^2f_1 + (-x-y)f_3$, $S(f_1, f_3) \xrightarrow{G}_+ 0 = h$.

Step 3.
Choose $\{f_2, f_3\}$.
$\mathcal{G} := \emptyset$
$S(f_2, f_3) = \frac{xy^3z}{xy^2z}(xy^2z) - \frac{xy^3z}{-xy^3}(-x^2y^2 - xy^3) = -x^2y^2z = -x(xy^2z) = -xf_2$.
Therefore, since $S(f_2, f_3) = -xf_2$, $S(f_2, f_3) \xrightarrow{G}_+ 0 = h$.

Thus, the algorithm ends. $G = \{f_1, f_2, f_3\}$ is our desired Gröbner basis.

Note that establishing a specific term order is important. Examples 1 and 2 above begin with the same two functions, yet the Gröber basis found in these two examples are distinct.

Example 3

Let $f_1 = 3x^2y + z$, $f_2 = xz + 2y \in \mathbb{Q}[x, y, z]$ and $z >_{lex} y >_{lex} x$, be the term order.

INITIALIZATION: $G := \{f_1, f_2\}$, $\mathcal{G} = \{\{f_1, f_2\}\}$

Step 1.
Choose $\{f_1, f_2\}$.
$\mathcal{G} := \emptyset$
$S(f_2, f_1) = \frac{xz}{xz}(xz + 2y) - \frac{xz}{z}(3x^2y + z) = 2y - 3x^3y = h$, which is reduced with respect to G since $lp(f_1) = z$, $lp(f_2) = xz$ and none of them divides h.
Since $h \neq 0$, let $f_3 := 2y - 3x^3y$.
$\mathcal{G} := \{\{f_1, f_3\}, \{f_2, f_3\}\}$
$G := \{f_1, f_2, f_3\}$

Step 2.
Choose $\{f_1, f_3\}$.
$\mathcal{G} := \{f_2, f_3\}$
$S(f_1, f_3) = \frac{yz}{z}(3x^2y + z) - \frac{yz}{2y}(2y - 3x^3y) = 3x^2y^2 + \frac{3}{2}x^3yz =$
$y(3x^2y + z) - \frac{1}{2}z(2y - 3x^3y) = yf_1 - \frac{1}{2}zf_3$. Therefore, since $S(f_1, f_3) = yf_1 - \frac{1}{2}zf_3$, $S(f_1, f_3) \xrightarrow{G}_+ 0 = h$.

Step 3.
Choose $\{f_2, f_3\}$.
$\mathcal{G} := \emptyset$
$S(f_2, f_3) = \frac{xyz}{xz}(xz + 2y) - \frac{xyz}{2y}(2y - 3x^3y) = 2y^2 + \frac{3}{2}x^4yz =$
$\frac{3}{2}x^3y(xz + 2y) + y(2y - 3x^3y) = \frac{3}{2}x^3yf_2 + yf_3$.
Therefore, since $S(f_2, f_3) = \frac{3}{2}x^3yf_2 + yf_3$, $S(f_2, f_3) \xrightarrow{G}_+ 0 = h$.

Thus, the algorithm ends. $G = \{f_1, f_2, f_3\}$ is our desired Gröbner basis.

5. References

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ABSTRACTS

Jeremy Ariche
Department of Mathematics
Morehouse College
Title: Methods for Finding Ramsey Numbers

A Ramsey number $R(s,t)$, where both s and t are positive integers, is defined as the lowest possible number of n vertices in a graph G where it contains a mutually adjacent sub-graph denoted as K_s and a mutually non-adjacent sub-graph denoted as K_t. It is understood that a Ramsey number can be bounded between two numbers. The process of finding a Ramsey number is difficult. After gaining a complete understanding of what Ramsey numbers are, one must begin to investigate methods of how previous Ramsey numbers were discovered. The proofs of them are rigorous. Once one understands how previous Ramsey numbers were discovered, one can begin to utilize similar methods. However, as s and t become larger, the methods become more difficult to utilize. The methods in this paper involve narrowing the upper and lower bounds of the Ramsey number until they are equal. The Ramsey number of interest is $R(5,5)$.

Victor Bailey
LSAMP Participant
Department of Mathematics
Georgia State University
Title: Sign Pattern matrices
Advisor: Dr. Zhongshan Li

A sign pattern matrix is a matrix whose entries are from the Set $\{+,-,0\}$. An $n \times n$ sign pattern matrix A requires all distinct eigenvalues if every real matrix whose entries have signs given by the corresponding entries of A, sign pattern represented by $sgn(A)$, has n distinct eigenvalues. For $n = 2$ and $n = 3$, all sign patterns requiring all distinct eigenvalues have been determined. However, the case of $n = 4$ is still not completely solved. We determine more necessary and/or sufficient conditions for 4×4 sign patterns to require all distinct eigenvalues.

Anthony J. Baker and Zachary L. Williams
Department of Mathematics, Sciences and Technology
Paine College
Title: Asymptotic Connectivity of Multihop Wireless Networks with the Log-normal Shadowing Model
Advisor: Dr. Lixin Wang

The unit-disk communication model of wireless networks is based on the path loss phenomenon alone and assumes that the received signal strength at a receiving node from a transmitting node is only determined by a deterministic function of the Euclidean distance between the two nodes. With randomly-deployed wireless networks under such a simple communication model, the vanishment of isolated nodes asymptotically implies connectivity of networks. However, in reality, the received signal strength often shows probabilistic variations induced by shadowing effects that are unavoidably caused by different levels of clutter (e.g., various background noises and obstructions) on the propagation path. To better capture physical reality, one should consider the variations of the received signal strength. It has been shown that a more accurate modeling of the physical layer is indeed important for better understanding of multihop wireless network characteristics. This generalized radio propagation model is referred to as the log-normal shadowing model which has been widely used by many researchers. In this paper we study the connectivity of multihop wireless networks with such a realistic model by giving necessary and sufficient conditions for the asymptotic vanishing of the isolated nodes in the wireless network under the log-normal shadowing model. The vanishing of isolated nodes is not only a prerequisite but also a good indication of network connectivity. All previous known work on network connectivity under such a model were obtained only based on simulation studies or ignoring the important boundary effect to avoid the challenging technical analysis, and thus cannot be applied to any practical wireless networks. It is extremely challenging to take the complicated boundary effect into consideration under such a realistic model because the transmission area of each node is an irregular region other than a circular area. To the best of our knowledge, there are no theoretical results obtained by rigorous analytical studies for connectivity of multihop wireless networks under such a realistic model when the boundary effect is taken into consideration. We assume the wireless nodes are represented by a Poisson point process with density n over a unit-area disk, and two nodes are directly connected if the received power at a receiving node under such a realistic model is not less than some given threshold. Under certain mild assumption, we derived necessary and sufficient conditions for the asymptotic vanishing of the isolated nodes in the wireless network under the log-normal shadowing model with the complicated boundary effect taken into consideration as the density n goes to infinity, and thus certain level of network connectivity can be expected when n is sufficiently large. The Palm theory for Poisson processes is utilized to derive the necessary and sufficient conditions for the asymptotic vanishing of the isolated nodes.

Curtis Clark
Department of Mathematics Morehouse College
Title: Integral Equations of Fredholm Type
Advisor: Dr. A. Brania

In this study, we investigate solutions of linear integral equations of Fredholm type with finite rank kernels. We then use kernels of finite rank to approximate solutions of linear integral equations of Fredholm Type with continuous kernels and give a relative error bound on the approximation. This method can be applied to approximate solutions of a larger class of linear integral equations of Fredholm type, namely those with L^2 kernels.

Joel Coppadge
Department of Mathematics
Morehouse College
Title: Fractional Differential Equations
Advisor: Dr. M. Sambandham

Using Caputo's fractional differential equation, we develop numerical methods. In particular for Caputo's fractional differential equation, we develop the Improved Euler method for fractional differential equations.

Jordan Culliver, **Courtney Mauck and Shawn Morris**
Department of Mathematics
Birmingham-Southern College
Title: A Mathematical Model for Solving Jigsaw Sudoku Puzzles

The Sudoku puzzle is the composite result of the study of magic squares, Latin squares and gerechte designs over multiple centuries. We provide a brief history of this study, as well as an outline of the rules of different variations of the puzzle, including both traditional and Jigsaw Sudoku. We explain how Arnold, Lucas, and Taalman showed that the Gröbner basis can be used to characterize the solution of a traditional Sudoku puzzle. Bartlett and Langville created an integer programming model that provides a solution to a traditional Sudoku puzzle by mathematically implementing the rules of the puzzle. By modifying a portion of this model to conform to the rules of Jigsaw Sudoku, we produce solutions to Jigsaw Sudoku puzzles.

Samuel Dillow and Jessica Gordon
Department of Mathematics and Statistics
Georgia State University
Title: The zero-forcing numbers of signed graphs
Advisor: Drs. Marina Arav and Hein van der Holst

In this paper we introduce the zero-forcing number of a signed graph. The motivation for introducing the zero-forcing number of a signed graph comes from the computation of the maximum nullity of a signed graph. For some classes of signed graphs, we determine the zero-forcing number.

Michael J. English and Kevin R. Bowman
Department of Mathematics
Morehouse College
Title: Product of Several Two-level Toeplitz Operators

In this paper we give necessary and sufficient conditions for when the product of two-level Toeplitz operators is again a Toeplitz operator. We then show that $T_f T_g = 0$ if and only if f or g is identically zero where T_f and T_g are two level Toeplitz operators. Finally, we conjectured zero product of three two level Toeplitz operators.

Michael Gray and Nicklaus Lynch
Department of Mathematics
Birmingham-Southern College
Title: A Mathematical Study on a Modified Lights Out Cube
Advisor: Dr. Bernadette Mullins

The Lights out game has been around since 1981 and mathematicians have talked about and analyzed it in almost every way. We looked outside the box by completely redesigning the game thereby making a completely new one. By rewriting the rules matrix as established by Todd Feil and Marlow Anderson in Turning Lights Out with Linear Algebra, we have found that one can create all types of new patterns in order to enhance the game. If you take sides off and move them then, more degrees of freedom open up for the game and allow more solvable sets and challenging patterns to arise. Our modified cube consists of taking the top face off of the cube and placed it inside the now open box along the diagonal of the cube. Following the rules of the original Lights Out game, our modified cube gave us new winning patterns and solutions.

Willtresca Heppard and Herns Mersamours
Department of Mathematics and Computer Science
Albany State University
Title: Solutions of Singularly Perturbed Third Order Ordinary Differential Equations
Advisor: Dr. Zephyrinus Okonkwo

Fluid flows through pipes and flat plates involve molecular motion which results in friction between the wall of the pipe and the body of the fluid. The fluid flow, coupled with viscosity of the fluid, result in the building of boundary layer of the fluid close to the pipe walls while lateral flow continues outside the boundary. Modeling of such fluid motion gives rise to singularly perturbed differential equations. In this work, we study singularly perturbed third order ordinary differential equations of the form

$$\epsilon\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

where $\epsilon > 0$ and $\epsilon \ll 1$ is a small parameter.

Kherri Hicks, **Erika Brown**, **Calhoun Marquis and Nathan Ketchup**
Department of Mathematics and Computer Science
Albany State University
Title: A Statistical Analysis of Network Penetration Testing in Information Assurance
Advisor: Dr. Robert Steven Owor

A network penetration test, colloquially called (PenTest), is a method of investigating the vulnerabilities of a computer system or network by simulating several attacks from malicious external and internal attackers. Potential vulnerabilities are investigated, detected and exploited to gain entry into the system. These vulnerabilities may be due to poor or improper system configuration, both known and unknown hardware or software flaws, and operational weaknesses in procedural or technical specifications and implementations. The analysis is carried out from a potential attackers point of view. Security flaws discovered during the process are presented to the organization in a report. Effective penetration testing combines this report with an accurate assessment of the potential impact to the organization, of different levels of attack. A comprehensive range of technical and procedural countermeasures to reduce risks is also recommended by the report. In this paper we propose a statistical methodology to identify and analyze a set of computer systems attack vectors by identifying higher-risk vulnerabilities that result from a combination of lower-risk vulnerabilities exploited in a possible set of sequences, discuss hidden vulnerabilities; estimate the magnitude of potential business and operational impacts of successful attacks, and assess the ability of network defenders to successfully detect and respond to the attacks. Finally, the paper suggests further areas of work in this new, exciting and challenging field.

Matthew Kelley
Department of Mathematics
Morehouse College
Title: Reactive oxygen species effect on the maturation of the CHRF megakaryocyte cell line
Advisors: Drs. Alaina Schlinker and William Miller (Northwestern University)

Megakaryocytes are cells located in the bone marrow that produce platelets. Producing platelets in vivo is a very efficient process but is extremely inefficient in vitro. To yield platelets, a megakaryocyte must have a high polyploidy level. In previous experiments done in the Miller lab with the erythro-megakaryocyte progenitor cell line K562, it has been shown that higher ploidy is directly proportional to higher intracellular reactive oxygen species levels. In this study, the effect of ROS was examined in the megakaryocyte progenitor cell line CHRF. CHRF cells were treated with the differentiating agent PMA and PMA+H2O2. These cells were examined for ploidy resulting in the conclusion that ROS does not play a role or increase the effects of polyploidization. In another experiment, intracellular ROS levels were examined upon varying the cells ability to adhere to the well surface. This showed that ROS production is severely impaired by an inability to adhere and spread. In the final experiment, the cells were stained with a DNA, ROS, and CD41+ stain for immunofluorescence. This showed that ROS is centralized inside the cell until proplatelet formation when the ROS spreads throughout the proplatelet extensions showing that ROS does play some role in proplatelet formation.

Melissa O. Miller
Department of Mathematics
Spelman College
Title: Queen Didos Area Problem

Queen Dido was challenged with the problem of maximizing area while keeping the perimeter constant. Problems of this sort are now called isoperimetric problems. We will mathematically model this problem and find a complete solution using the Calculus of Variations. The Calculus of Variations is a generalization of the ordinary Calculus involving numbers, to a Calculus involving functions and curves.

Sarah Minion and Pritul Patel
Department of Mathematics
Clayton State University
Title: On The Edge-Balanced Index Set of $K_{13,7}$
Advisors: Drs. Elliot Krop and Christopher Raridan

Let G be a graph with vertex set V and edge set E. Consider a binary labeling of E, called an edge-friendly labeling, where the absolute difference in the number of edges labeled 1 and 0 is no more than one. When the degree of every vertex is odd, an edge-friendly labeling induces a binary labeling on the vertices as follows: a vertex that is incident with more 1-edges than 0-edges is labeled 1; otherwise, the vertex is labeled 0. The edge-balanced index set of G is the collection of the absolute differences in the number 1-vertices and the number of 0-vertices over all edge-friendly labellings of G. In this paper, we determine the edge-balanced index set of the complete bipartite graph $K_{13,7}$ as a next step toward a general solution to a problem introduced by Kong, Wang, and Lee and later examined by Krop and Sikes.

Michael Ngo, **Marcus Bartlet**, **Thuhong Nguyen and Petra President**
Department of Mathematics
Clayton State University
Title: Small Pattern Gallai Ramsey Numbers
Advisor: Dr. Elliot Krop

The minimum order of any complete graph so that for any coloring of the edges by k colors it is impossible to avoid a monochromatic or rainbow triangles is known as the smallest Gallai-Ramsey number. For any graph H with edges colored from the above set of k colors, if we consider the condition of excluding H in the above definition, we produce a *pattern Gallai-Ramsey number*. In this talk, we consider this problem when H is a two-colored cycle with two consecutive colors and discuss the solution in terms of k.

Laura Parrish
Department of Mathematics
Clayton State University
Title: On Opinionated Complete Bipartite Graphs
Advisors: Drs. Christian Barrientos and Elliot Krop

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and f be a $0-1$ labeling of $E(G)$ so that the absolute difference in the number of edges labeled 1 and 0 is no more than one. Call such a labeling f *edge-friendly*. We say an edge-friendly labeling induces a *partial vertex labeling* if vertices which are incident to more edges labeled 1 than 0, are labeled 1, and vertices which are incident to more edges labeled 0 than 1, are labeled 0. Vertices that are incident to an equal number of edges of both labels we call *unlabeled*. We call an edge-friendly labeling of a graph *opinionated*, if given the above definition, every vertex receives a label. We show that for every positive integers n, m, $K_{n,m}$ is opinionated.

Pritul Patel
Department of Mathematics
Clayton State University
Title: On the Edge-Balanced Index Sets of Complete Odd Bipartite Graphs
Advisor: Dr. Christopher Raridan

We determine the edge-balanced index sets for all complete bipartite graphs with parts of odd order.

Aaron Reaves
Department of Mathematics
Morehouse College
Title: Analysis of a System of Non-linear Boundary Value Problems with Lidstone Boundary Conditions
Advisor: Dr. Tuwaner Lamar

This project discusses the existence of positive solutions for a system of bending elastic beam equations with Lidstone boundary conditions by using

xed point theory. In mechanics, the problem describes the deformation of elastic beams in equilibrium state, whose two ends are simply supported. This paper will establish the existence of solutions for three beams and then attempt to generalize the results for some n number of beams.

Zachary Richards, **Aaron Sherrill and Jean Damascene Rugamba**
Department of Mathematics
Birmingham-Southern College
Title: Extending the Mathematical Study of Bracketology
Advisor: Dr. Bernadette Mullins

Bracketology is the process of predicting the results of the NCAA Division I mens basketball tournament and filling out a bracket based on the predictions. There are different mathematical methods to take on this prediction. A heavily studied formula for bracketology is the Colley Method. Our study, motivated by Tim Chartiers paper, Bracketology: How Can Math Help, focuses on extending the Colley Method to construct a more accurate predictive tool for the tournament. We carried out this extension by incorporating three factors: away winning percentage minus home losing percentage, strength of schedule, and a time-based game weight. While adjusting the three factors, our extended Colley method based formula was used to calculate predictive brackets for the 2009 tournament. Out of our various methods tested, the weekly time-based game weight paired with a scaling of our remaining two factors gave us our best predictive bracket for the tournament.

Zollie White
Mentor: Dr. Benedict Nmah
Department of Mathematics
Morehouse College
Title: Calculating Gröbner Bases in a Polynomial Ring
Advisor: Dr. B. Nmah

The concept of **Gröbner basis** is relatively new. Though Bruno Buchberger first introduced the concept of Gröbner basis in his doctoral thesis in 1965, the concept remained unknown to the research community until a decade later. The theory of Gröbner basis has become an important subarea in computer algebra. It is included in all the major program systems of symbolic computation, and it is being fruitfully applied to a variety of research areas. Having developed a basic understanding of what a Gröbner basis is, I calculated Gröbner bases over given polynomial rings using the Multivariate Division Algorithm.