Proceedings of the Harriett J. Walton Symposium on Undergraduate Mathematics Research

Volume 10

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Preface

The Department of Mathematics initiated the Harriett J. Walton Symposium on Undergraduate Mathematics Research to encourage undergraduate students in mathematics research and practice. We believe that undergraduate research experiences must count among the most challenging and rewarding experiences for college students. To research and present material beyond what is traditionally covered in the classroom to explore mathematics independently may be considered the best career preparation for students regardless of their post-college plans. This yearly symposium on undergraduate research is dedicated to Professor Harriett J. Walton who served forty-two years on the faculty of Morehouse College.

The Sixth Annual Harriett J. Walton Symposium on Undergraduate Mathematics Research was held on the campus of Morehouse College in Atlanta, Georgia, USA on April 5, 2008. Eighteen students from Albany State University, Clayton State University, Fort Valley State University, Georgia State University, Morehouse College, Spelman College, and Southern Polytechnic State University gave presentations on their research and studies in mathematics and related fields. The Symposium was sponsored by the Department of Mathematics and the Division of Science and Mathematics of Morehouse College through the generous support of The Mathematical Association of America (MAA) Regional Undergraduate Mathematics Conference Program through National Science Foundation Grant DMS-0241090. This volume contains four articles and twelve abstracts submitted by the Symposium participants and their advisors.

The organizers of the Symposium thank the presenters and their advisors for preparing a remarkable collection of lectures for the Symposium. We thank the referees for their service to evaluate and improve the papers before their publication. We thank the administration of Morehouse College for their generous support, especially Dr. J. K. Haynes, Dean of the Division of Science and Mathematics, Dr. John Williams, Provost and Senior Vice President for Academic Affairs, and President Dr. Robert M. Franklin. We thank the MAA for their support, advice, and materials for the Symposium. Special thanks to Ms. Sandra Strickland for coordinating many aspects of the Symposium. Finally we thank Professor Walton for attending the Symposium.

Professor Harriett J. Walton

In September 1958, Harriett J. Walton joined the faculty of Morehouse College during the presidency of Benjamin Elijah Mays. She became a member of a team of three persons in the Department of Mathematics where she worked with the legendary Claude B. Dansby who served as Department Chair. Dr. Walton and her two colleagues taught all of the mathematics for the majors as well as the mathematics for non-science students. Dr. Walton relates that two of her favorite courses that she taught during this period were Abstract Algebra and Number Theory. The three member mathematics department did an excellent job of preparing their mathematics majors for graduate school and the other students for success in their respective disciplines. In fact it was during this period of history that Morehouse gained the reputation of being an outstanding Institution especially for African American men. As the department grew, Dr. Walton shifted her attention away from mathematics majors and began to concentrate on students who needed special attention and care in order to succeed in mathematics. She became an advisor, mentor, tutor and nurturer to a large number of students matriculating at Morehouse College. Because of the caring attitude that she had for her students, some of them to this day refer to her as “Mother Walton”.

Dr. Walton has never been satisfied with mediocrity. Throughout her teaching career she demonstrated a love for learning. In 1958 when she arrived at Morehouse College she had an undergraduate degree in mathematics from Clark College in Atlanta, Georgia, a Master of Science degree in mathematics from Howard University, Washington D.C., and a second Master's degree in mathematics from Syracuse University. While at Morehouse teaching full time and raising a family of four children, Dr. Walton earned the Ph.D. degree in Mathematics Education from Georgia State University. After receiving her doctorate, Dr. Walton realized the emerging importance of the computer in education so she returned to school and in 1989 earned a Master’s degree in Computer Science from Atlanta University. She is indeed a remarkable person.

Dr. Walton’s list of professional activities, awards and accomplishments during her career is very impressive and too lengthy to be enumerated here. However a few special ones are her memberships in Alpha Kappa Mu, Beta Kappa Chi, Pi Mu Epsilon, and the prestigious Phi Beta Kappa Honor Society. Additionally she was selected as a Fulbright Fellow to visit Ghana and Cameroon in West Africa. Dr. Walton’s professional memberships included the American Mathematical Society, the Mathematical Association of America, National Council of Teachers of Mathematics (NCTM) and the National Association of Mathematicians (NAM). She served as Secretary/Treasurer of NAM for ten years. In May 2000, Dr. Walton retired from Morehouse College after forty-two (42) years of service.

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The Role of the Vandermonde Matrix in Hermite Interpolatory Polynomials

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Abstract: In his paper, "The Generalized Vandermonde Matrix", Kalman [2], states the Vandermonde determinant of order n as an identity. In this paper I use the Principle of Mathematical Induction to establish the identity. I also discuss the role of the Vandermonde Matrix in Hermite interpolatory polynomials.

1. Introduction

A Vandermonde matrix of size $n \times n$ is completely determined by n arbitrary numbers $x_i \in \mathbf{R}$ for $i = 1, 2, \cdots, n$, in terms of which its n^2 elements are the integer powers x_i^{j-1}, where $i, j = 1, 2, \cdots, n$. In the literature, there are two possible forms, depending on whether the $i's$ are viewed as rows and the $j's$ as columns, or vice versa. If we view the $i's$ as rows and the $j's$ as columns, we get a Vandermonde system that is a linear system of equations of the form

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}^T \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, \tag{1.1}$$

where the system (1.1) solves for the unknown coefficients, c_i, which fits a polynomial to the $n-$pairs (x_i, y_i) of abscissas and ordinates.

On the other hand, if we view the $i's$ as rows and the $j's$ as columns, we get a Vandermonde system [1] that is a linear system of equations of the form

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}, \tag{1.2}$$

where given the values of n points x_i, the system (1.2) solves for the unknown weights, w_i, assigned in order to match the given values q_j of the first n moments.

The $n \times n$ coefficient matrices in (1.1) and (1.2) are Vandermonde matrices, so named after Alexandre-Theophile Vandermonde who is accredited as being the "founder of the theory of determinants" [3]. Vandermonde matrices are well-known; they are ill-conditioned matrices in the sense that they are "nearly singular" and can readily become singular if some of their entries are changed ever so slightly. The Vandermonde matrices occur in many approximation and interpolation problems (1.1), the construction, and also in other contexts (e.g. differential equations, initial value problems, and recursively defined sequences).

From elementarty linear algebra, we know that the determinant of a matrix is invariant under matrix transposition. Thus, without loss of generality, we will use the coefficient matrix in (1.2) for our purpose.

2. Calculating Vandermonde Determinant for $n = 3$ and 4

From the introduction, we will calculate the determinant [1] of the 3×3 and 4×4 Vandermonde matrix respectively, in preparation for the inductive proof to be presented in section 4.

$$V = V(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}$$

is a 3×3 Vandermonde matrix. Now,

$$\det V = |V(x_1, x_2, x_3)| = 1 \begin{vmatrix} x_2 & x_3 \\ x_2^2 & x_3^2 \end{vmatrix} - 1 \begin{vmatrix} x_1 & x_3 \\ x_1^2 & x_3^2 \end{vmatrix} + 1 \begin{vmatrix} x_1 & x_2 \\ x_1^2 & x_2^2 \end{vmatrix}$$

$$\Longleftrightarrow \det V = x_2x_3^2 - x_3x_2^2 - x_1x_3^2 + x_3x_1^2 + x_1x_2^2 - x_2x_1^2 \tag{2.1}$$

We rewrite the determinant in eq.(2.1):

$\det V = x_3\left[x_2x_3 - x_2^2 - x_1x_3 + x_1^2\right] + x_1x_2^2 - x_2x_1^2 = x_3\left[x_3(x_2 - x_1) - \left(x_2^2 - x_1^2\right)\right] + x_1x_2^2 - x_2x_1^2$

where we simplify the det V below:

$x_3\left[x_3(x_2 - x_1) - (x_2 - x_1)(x_2 + x_1)\right] + \left(x_1x_2^2\right) - \left(x_2x_1^2\right) = (x_2 - x_1)\left[x_3^2 - (x_2x_3) - (x_3x_1) - (x_2x_1)\right].$

Using the previously derived equations, we have:

$\det V = (x_2 - x_1)\left[x_3(x_3 - x_2) - x_1(x_3 - x_2)\right] = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1) = \prod_{1 \le i < j \le 3}(x_j - x_i).$

Next, for $n = 4$, we have the following 4×4 Vandermonde Matrix:

$$V = V(x_1, x_2, x_3, x_4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix}.$$

We need to verify that

$$\det V = \prod_{1 \le i < j \le 4}(x_j - x_i) \tag{2.2}$$

We will now compute det V as follows:

$$\det V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} =$$

$$= \begin{vmatrix} x_2 & x_3 & x_4 \\ x_2^2 & x_3^2 & x_4^2 \\ x_2^3 & x_3^3 & x_4^3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 & x_4 \\ x_1^2 & x_3^2 & x_4^2 \\ x_1^3 & x_3^3 & x_4^3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_4 \\ x_1^2 & x_2^2 & x_4^2 \\ x_1^3 & x_2^3 & x_4^3 \end{vmatrix} - \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix}.$$

The above simplifies to:

$$\begin{aligned} & x_2\left[x_3^2x_4^3 - x_3^3x_4^2\right] - x_3\left[x_2^2x_4^3 - x_2^3x_4^2\right] + x_4\left[x_2^2x_3^3 - x_2^3x_3^2\right] - \\ & -x_1\left[x_3^2x_4^3 - x_3^3x_4^2\right] + x_3\left[x_1^2x_4^3 - x_1^3x_4^2\right] - x_4\left[x_1^2x_3^3 - x_1^3x_3^2\right] + \\ & +x_1\left[x_2^2x_4^3 - x_2^3x_4^2\right] - x_2\left[x_1^2x_4^3 - x_1^3x_4^2\right] + x_4\left[x_1^2x_2^3 - x_1^3x_2^2\right] - \\ & -x_1\left[x_2^2x_3^3 - x_2^3x_3^2\right] + x_2\left[x_1^2x_3^3 - x_1^3x_3^2\right] - x_3\left[x_1^2x_2^3 - x_1^3x_2^2\right] = \\ & = x_2x_3^2x_4^3 - x_2x_3^3x_4^2 - x_3x_2^2x_4^3 + x_3x_2^3x_4^2 + \\ & +x_4x_2^2x_3^3 - x_4x_2^3x_3^2 - x_1x_3^2x_4^3 + x_1x_3^3x_4^2 + \\ & +x_3x_1^2x_4^3 - x_3x_1^3x_4^2 - x_4x_1^2x_3^3 + x_4x_1^3x_3^2 + \\ & +x_1x_2^2x_4^3 - x_1x_2^3x_4^2 - x_2x_1^2x_4^3 + x_2x_1^3x_4^2 + \\ & +x_4x_1^2x_2^3 - x_4x_1^3x_2^2 - x_1x_2^2x_3^3 + x_1x_2^3x_3^2 + \\ & +x_2x_1^2x_3^3 - x_2x_1^3x_3^2 - x_3x_1^2x_2^3 + x_3x_1^3x_2^2 \end{aligned}$$

which by adding some redundant terms factors as

$$\underbrace{(x_2 - x_1)(x_3 - x_2)(x_3 - x_1)}_{\prod_{1\le i<j\le 3}(x_j - x_i)}\underbrace{(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)}_{\prod_{i=1}^{3}(x_4 - x_i)}$$

$$= \prod_{1\le i<j\le 3}(x_j - x_i)\prod_{i=1}^{3}(x_4 - x_i) = \prod_{1\le i<j\le 4}(x_j - x_i).$$

3. The Vandermonde Determinant

In this section, we will use the principle of mathematical induction to show that for all positive integers n, we have

$$|V(x_1, x_2, \cdots, x_n)| = \prod_{1\le i<j\le n}(x_j - x_i).$$

Proof. Since this statement holds trivially where $x_i = x_j$ for some $i \neq j$, we suppose $x_j \neq x_i$ whenever $i \neq j$. For $n = 3$, by (2.1) we have

$$\det V = |V(x_1, x_2, x_3)| = \prod_{1\le i<j\le 3}(x_j - x_i).$$

Now assume
$|V(x_1,x_2,\cdots,x_k)| = \prod_{1\le i<j\le k}(x_j-x_i) = (x_2-x_1)(x_3-x_2)(x_3-x_1)\cdots(x_k-x_1)\cdots(x_k-x_{k-1})$.

From (3.2) and the inductive hypothesis (4.1), we have

$$\begin{aligned}
&|V(x_1,x_2,\cdots,x_{k+1})| \\
&\quad = \underbrace{(x_2-x_1)(x_3-x_2)(x_3-x_1)\cdots(x_k-x_1)\cdots(x_k-x_{k-1})}_{\prod_{1\le i<j\le k}(x_j-x_i)}\underbrace{(x_{k+1}-x_1)\cdots(x_{k+1}-x_k)}_{\prod_{i=1}^{k}(x_{k+1}-x_i)} \\
&\quad = \prod_{1\le i<j\le k}(x_j-x_i)\prod_{i=1}^{k}(x_{k+1}-x_i) = \prod_{1\le i<j\le k+1}(x_j-x_i).
\end{aligned}$$

Hence, for all $n\in\mathbf{N}$, $\det V(x_1,\cdots,x_n) = \prod_{1\le i<j\le n}(x_j-x_i)$. □

4. Hermite Interpolatory Polynomials

The Lagrange interpolating polynomial is the polynomial of degree less than or equal to $n-1$ and is given by

$$P(x) = \sum_{j=1}^{n} P_j(x),$$

where $P_j(x) = y_j \prod_{k=1}^{n} \frac{x-x_j}{x_j-x_k}$. (see [3]).

A Taylor polynomials of degree n matches the function and its first n-th derivatives at a single point. On the other hand, a Lagrange Polynomial of degree n matches the function values at $n+1$ points. Thus, the question had been whether the ideas of Taylor and Lagrange could be used to get an interpolatory polynomial that would match both the function values and some number of derivatives at multiple points. The answer is YES! Such polynomials exist and they are referred to as osculating polynomials.

Given $n+1$ distinct points $x_0,x_1,\dots,x_n$ with $x_i\in[a,b]$ and $n+1$ non-negative integers $m_0,m_1,\dots,m_n$, let $m=\max\{x_i\}$, for $0\le i\le n$. The osculating polynomial approximation of a function f that is continually differentiable on $[a,b]$ at x_i is the polynomial (of lowest possible order) that agrees with $\{f(x_i)\}_{i=0}^{n}$ for every i. The degree of the osculating polynomial is at most $M = n+\sum_{i=0}^{n} m_i$. Hermite Interpolatory polynomials [3] are osculating polynomials of degree $n+1$ (i.e. $\sum_{i=0}^{n} m_i = 1$).

If $f\in C^1[a,b]$ and $x_0,\dots,x_n\in[a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0,\cdots,x_n$ is a polynomial of degree at most $2n+1$ given by:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x),$$

Where $H_{n,j}(x) = \left[1-2(x-x_j)L'_{n,j}(x_J)\right]L^2_{n,j}(x)$ and $\hat{H}_{n,j}(x) = (x-x_j)L^2_{n,j}(x)$. See [1]. In this context, $L_{n,j}$ denotes the j-th Lagrange coefficient polynomial of degree n.

One of the primary applications of Hermite interpolatory polynomials is in the development of Gaussian quadrature for numerical integration. The most commonly seen Hermite interpolatory polynomial is the cubic one which satisfies,

$$H_3(x_0) = f(x_0), H_3'(x_0) = f'(x_0),$$

and

$$H_3(x_1) = f(x_1), H_3'(x_1) = f'(x_1).$$

Writing it explicitly we have:

$$\begin{aligned} H_3(x) = & \left[1 + \frac{2(x-x_0)}{x_1-x_0}\right]\left[\frac{(x_1-x)}{x_1-x_0}\right]^2 f(x_0) + (x-x_0)\left[\frac{(x_1-x)}{x_1-x_0}\right]^2 f'(x_0) + \\ & + \left[1 + \frac{2(x_1-x)}{x_1-x_0}\right]\left[\frac{(x-x_0)}{x_1-x_0}\right]^2 f(x_1) + (x-x_1)\left[\frac{(x-x_0)}{x_1-x_0}\right]^2 f'(x_1). \end{aligned}$$

Here is an example: Use Hermite Interpolation to find an approximation to $f(2.5)$, given the following data:

TABLE 1

k	x_k	$f(x)$	$f'(x)$
0	2.2	.5207843	-.0014878
1	2.4	.5104147	-.1004889
2	2.6	.4813306	-.1883635

First we compute the Lagrange polynomials and their derivatives:

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \tfrac{25}{2}x^2 + \tfrac{125}{2}x + 78,$$

$$L_{2,0}'(x) = 25x + \tfrac{125}{2},$$

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -25x^2 + 120x + 143,$$

$$L_{2,1}'(x) = -50x + 10,$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \tfrac{25}{2}x^2 + 115x + 66,$$

and

$$L_{2,2}'(x) = 25x + 115.$$

The polynomials $H_{2,j}$ and $\hat{H}_{2,j}$ are then:

$$H_{2,0}(x) = \left[1 - 2(x-x_0)L_{2,0}'(x_0)\right]L_{2,0}^2(x) = (-235x - 516)\left[\tfrac{25}{2}x^2 + \tfrac{125}{2}x + 78\right]^2,$$

$$H_{2,1}(x) = \left[1 - 2(x-x_1)L_{2,1}'(x_1)\right]L_{2,1}^2(x) = (-220x - 527)\left[-25x^2 + 120x + 143\right]^2,$$

$$H_{2,2}(x) = \left[1 - 2(x-x_2)L_{2,2}'(x_2)\right]L_{2,2}^2(x) = (-360x - 935)\left[\tfrac{25}{2}x^2 + 115x + 66\right]^2,$$

$$\hat{H}_{2,0}(x) = (x-x_0)L_{2,0}^2(x) = (x-2.2)\left[\tfrac{25}{2}x^2 + \tfrac{125}{2}x + 78\right]^2,$$

$$\hat{H}_{2,1}(x) = (x-x_1)L_{2,1}^2(x) = (x-2.4)\left[-25x^2 + 120x + 143\right]^2,$$

$$\hat{H}_{2,2}(x) = (x-x_2)L_{2,2}^2(x) = (x-2.6)\left[\tfrac{25}{2}x^2 + 115x + 66\right]^2,$$

Finally, since $n = 2$, and $x = 2.5$, the expression

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

becomes

$$\begin{aligned} H_5(2.5) = {} & .5207843H_{2,0}(2.5) + .5104147H_{2,1}(2.5) + .4813306H_{2,2}(2.5) \\ & -.0014878\hat{H}_{2,0}(2.5) - .1004889H_{2,1}(2.5) - .1883635H_{2,2}(2.5) \end{aligned}$$

which implies that

$$\begin{aligned} H_5(2.5) = .5207843H_{2,0}(2.5) + .5104147H_{2,1}(2.5) + .4813306H_{2,2}(2.5) \\ -.0014878\hat{H}_{2,0}(2.5) - .1004889H_{2,1}(2.5) - .1883635H_{2,2}(2.5). = .4948112 \end{aligned}$$

From the table, we see that f is decreasing on $[2.2, 2.6]$ since $f'(x) < 0$ for $x \in [2.2, 2.6]$. Thus, we expect $f(2.4) < f(2.5) < f(2.6)$. Hence we expect $f(2.4) < H(2.5) < f(2.6)$ because $H(x)$ matches $f(x)$ on $[2.2, 2.6]$ so $H(2.5) = .4948112$ is acceptable.

The Vandermonde matrix corresponding to the previous example where $x_0 = 2.2, x_1 = 2.4$, and $x_2 = 2.6$ is

$$V = V(x_0, x_1, x_2) = \begin{bmatrix} 1 & 1 & 1 \\ 2.2 & 2.4 & 2.6 \\ 4.84 & 5.76 & 6.76 \end{bmatrix}.$$

From (3.1), we have

$$\begin{aligned} \det V = (x_1 - x_0)(x_2 - x_1)(x_2 - x_0) = (2.4 - 2.2)(2.6 - 2.4)(2.6 - 2.2) = \\ = (.2)(.2)(.4) = .016. \end{aligned}$$

5. Summary

In summary, we have established that a given Vandermonde matrix of size $n \times n$, $V = V(x_1, x_2, \cdots, x_n)$, we have $\det V = \prod_{1 \le i < j \le n} (x_j - x_i)$, where $x_i \in \mathbf{R}$ for $i = 1, 2, \cdots, n$.

We also demonstrated how the determinant of the Vandermonde matrix can be used to determine the solvability of a Vandermonde system using Hermite interpolatory polynomials. This project has taught me among other things, how mathematicians often come up with some of their research topics.

6. Acknowledgments

I would like to thank Dr. Benedict Nmah for his willingness to serve as my project advisor, and for all his help, guidance and encouragement throughout the project. I would like to acknowledge Dr. Chuang Peng, Dr. Duane Cooper, and the rest of the Mathematics Department faculty at Morehouse College for helping me either vicariously or otherwise throughout my studies at Morehouse College. Finally, I would like to also acknowledge my fellow classmates and peers for their critiques and support.

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Analysis of Lorenz System and Data Assimilation Techniques

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Abstract: Edward Norton Lorenz (May 23, 1917 - April 16, 2008) was an American mathematician, meteorologist, and a pioneer of chaos theory. He discovered the strange attractor notion and coined the term butterfly effect. He also studied two-dimensional convection in a horizontal layer of fluid heated from below. These studies resulted in the Lorenz System (see [1, 2]). The Lorenz System is made up of three ordinary differential equations:

$$\frac{dx}{dt} = \sigma(y - x), \ \frac{dy}{dt} = \rho x - y - xz, \ \frac{dz}{dt} = xy - \beta z$$

where σ, ρ, β are parameters. In this research, these equations will be examined to find the fixed (critical) points of the Lorenz System, determine the stability and instability of the fixed points, calculate the numerical solutions of the Lorenz System using a computer program called Matlab, and applying data assimilation to the Lorenz System.

1. Introduction

In 1963, while conducting an experiment in the theory of forecasting, Edward Lorenz had come across a phenomenon that later came to be known as "chaos". When Lorenz delivered "The Essence of Chaos" in 1990 at the University of Washington, he broke it up into three distinct parts. First he defined chaos and illustrated its basic properties with some simple examples, and ended by describing some related phenomena-nonlinearity, complexity, and fractal nature- that had also come to be called "chaos". Next he dealt with the global weather as a complicated example of a chaotic system. Lastly, he presented an account of our growing awareness of chaos, offered a prescription via which one could design ones own chaotic systems, and end with some philosophical speculations.

Similar in phenomena to the Fourier system to explain the passage of heat through a metal bar, the Lorenz system explains the path of the heating and cooling currents involved in convection. The ordinary differential equations presented previously represent the change in the intensity of convection motion with respect to time ($\frac{dx}{dt}$), the change in the temperature difference between ascending and descending currents with respect to time ($\frac{dy}{dt}$), and the change in the distortion in the vertical temperature profile with respect to time ($\frac{dz}{dt}$).

The passage of the currents revolve about certain fixed points, also referred to as critical points. As in elementary mathematics when one found the critical points of a polynomial by setting the derivative equation equal to zero, the critical points of the Lorenz system are found by setting the differential equations equal to zero and solving for the values of x, y, and z. This process is shown later in this document. Involved in the Lorenz system are three parameters, σ, ρ, and β. These parameters control the locations of the critical points and the movement of the currents, as can be seen in the presentation of these currents in

Matlab. Once the critical points are found it is important to take note of how stable the currents are at those points. This is done by analyzing the equations near those points. This will also be demonstrated later in the project.

2. Preliminaries

The Lorenz system, as was said before, is a set of ordinary differential equations, dealing with variables $x(t)$ (the intensity of convection motion), $y(t)$ (the temperature difference between ascending and descending currents), and $z(t)$ (the distortion in the vertical temperature profile). In mathematics, an ordinary differential equation (ODE) is an equation in which there is only one independent variable and one or more derivatives of a dependent variable with respect to the independent variable, so that all the derivatives occuring in the equation are ordinary derivatives. The derivatives represent the instantaneous slope at a given point on the graph of the equation. That is while the slope, m, with respect to time, t, of a secant line of the graph, intersecting at points (x_1, t_1) and (x_2, t_2) can be given by $m = \frac{x_1 - x_2}{t_1 - t_2}$, if the limit of this equation is taken as x_1 approaches x_2 and t_1 approaches t_2 you get $\frac{dx}{dt}$, known as the ordinary derivative of the graph in the x direction with respect to t.

These derivatives are key in finding the critical points of the Lorenz system. These critical points are the places where the derivative does not exist and where the graph of the solutions to the Lorenz system tends towards. Some of the calculations used to derive the critical points involve knowing a little complex analysis. Namely Euler's Formula which states for any real number, x, $e^{ix} = \cos(x) + i\sin(x)$, where $i = \sqrt{-1}$ is called an imaginary, or complex, number. Using this and some other calculations to be detailed later in this paper, we find the critical points to be in terms of three positive parameters. These parameters are β, σ, and ρ which represent a set physical qualifiers associated with the fluid. Here, σ is known as the Prandtl number, ρ is the Rayleigh number, and β is the geometric factor.

In this research project, matrix algebra is used. It is therefore important to know that a system of equations can be represented as a matrix. That is, for example a matrix with three rows and three columns known as a (3×3) matrix. One can express the equation:

$$\begin{aligned} ax + by + cz &= b_1 \\ dx + ey + fz &= b_2 \\ gx + hy + iz &= b_3 \end{aligned}$$

in matrix form as:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

A special type of matrix is the $(n \times n)$ identity matrix of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In order to multiply two matrices the number of columns of the first has to be equal to the number of rows in the second, and the resultant matrix has the same number of rows as the first and the same number of columns as the second. This is because the process of multiplying matrices is entries in rows of the first matrix times corresponding entries in the corresponding columns of the second then adding the products for each entry in the resultant matrix. For example

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \text{ x } \begin{pmatrix} j \\ m \\ p \end{pmatrix} = \begin{pmatrix} aj + bm + cp \\ dj + em + fp \end{pmatrix}$$

The last piece of matrix algebra needed is to understand the determinant of a (3×3) matrix. With a (3×3) matrix only, the determinant can be determined by adding the products of the downward diagonals from left to right and subtracting the downward diagonals from right to left. That is

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a*(e*i - h*f) - b*(d*i - g*f) + c*(d*h - g*e)$$

where $*$ represents multiplication of two real numbers.

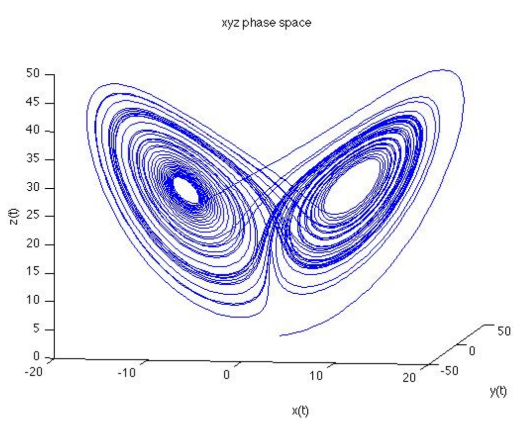
3. Critical Points and Stability Analysis

Lorenz studied two-dimensional convection in a horizontal layer of fluid heated from below. Analysis from the study of this problem resulted in the derivation of the following set of coupled ordinary differential equations:

$$\frac{dx}{dt} = \sigma(y - x), \ \frac{dy}{dt} = \rho x - y - xz, \text{ and } \frac{dz}{dt} = xy - \beta z,$$

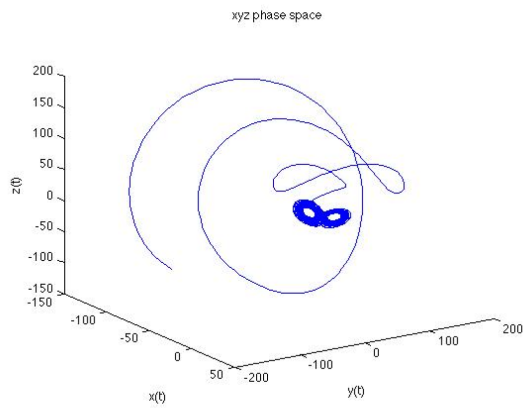
where $\frac{dx}{dt}$ is the change in the intensity of convection motion over time, $\frac{dy}{dt}$ is the change in temperature difference between ascending and descending currents over time, and $\frac{dz}{dt}$ is the change in the vertical temperature profile over time. The parameters, σ, ρ,and β, are all positive values associated with the system. σ is known as the Prandtl number, ρ is the Rayleigh number, and β is the geometric factor.

To demonstrate the chaotic behavior we set $\sigma = 10, \rho = 28$, and $\beta = \frac{8}{3}$.



The critical points are the points where the derivative vanishes. Keeping in mind $\sigma, \rho, \beta > 0$, these critical points can be found by setting the appropriate derivatives equal to zero:

$$\frac{dx}{dt} = 0, \ \frac{dy}{dt} = 0, \ \frac{dz}{dt} = 0.$$



This results in the following set of algebraic equations:

$$\sigma(y-x)=0, \implies y=x; \qquad xy-\beta z=0, \implies z=\frac{x^2}{\beta}$$

$$-xz+\rho x-y=0, \implies x\left(-\frac{x^2}{\beta}+\rho-1\right)=0, \implies x=\pm\sqrt{\beta(\rho-1)}.$$

where $\beta(\rho-1)>0$ so that this value will be a real number. Since both y and z are in terms of x and x is independent of the other variables, x determines the critical points. That is where $x=0$ the critical point is $(0,0,0)$, where $x=\sqrt{\beta(\rho-1)}$ the critical point is at $(\sqrt{\beta(\rho-1)},\sqrt{\beta(\rho-1)},\rho-1)$, and lastly where $x=-\sqrt{\beta(\rho-1)}$ the critical point is at $(-\sqrt{\beta(\rho-1)},-\sqrt{\beta(\rho-1)},\rho-1)$.

The reason why you only see two critical points in the above figure is due to the fact that the initial starting point is at $(1,1,1)$ so point $(0,0,0)$ is not included. To show the third point the figure below starts at $(-100,-100,-100)$, but it is important to know that since this deals with time and you can not have negative time this has no physical applications.

When analyzing the stability of a critical point, (x_0,y_0,z_0), it is important to consider how the equations $(x(t),y(t),z(t))$ behave near (x_0,y_0,z_0). This is done by setting $x(t)=x_0+\delta x(t)$, $y(t)=y_0+\delta y(t)$, and $z(t)=z_0+\delta z(t)$ (for $|\delta x(t)|$, $|\delta y(t)|$, and $|\delta z(t)|$ very very small), substituting them into the original differential equations and solving for $\frac{d(\delta x)}{dt}$, $\frac{d(\delta y)}{dt}$, and $\frac{d(\delta z)}{dt}$ respectively. That is we set

$$\frac{dx}{dt}=\frac{d}{dt}(x_0+\delta x)=\frac{d(\delta x)}{dt}=\sigma(y_0+\delta y-x_0-\delta x)=\sigma(y_0-x_0)+\sigma(\delta y-\delta x)$$

thus, since $y=x$

$$\frac{dx}{dt}=\sigma(\delta y-\delta x).$$

Likewise

$$\frac{dy}{dt}=\frac{d(\delta y)}{dt}=(\rho-z_0)\delta x-\delta y-x_0\delta z$$

and

$$\frac{dz}{dt}=\frac{d(\delta z)}{dt}=y_0\delta x+x_0\delta y-\beta\delta z.$$

Note that since $|\delta x(t)|$, $|\delta y(t)|$, and $|\delta z(t)|$ are very very small the product of any combination of them is roughly 0. Now if we set:

$$\mathbf{P}(t)=\begin{pmatrix}\delta x(t)\\ \delta y(t)\\ \delta z(t)\end{pmatrix}$$

and we can write $\frac{d\mathbf{P}(t)}{dt}$ in terms of the product of matrix A and vector $\mathbf{P}(t)$:

$$\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z_0 & -1 & -x_0 \\ y_0 & x_0 & -\beta \end{pmatrix} \begin{pmatrix} \delta x(t) \\ \delta y(t) \\ \delta z(t) \end{pmatrix} = A\mathbf{P}(t)$$

If we let $\mathbf{P}(t) = \mathbf{p}e^{\lambda t}$, where $\mathbf{p}$ is a constant vector and λ is a scalar, $\frac{d\mathbf{P}(t)}{dt} = \frac{d}{dt}(\mathbf{p}e^{\lambda t}) = \mathbf{p}\lambda e^{\lambda t} = A\mathbf{P}(t) = A\mathbf{p}e^{\lambda t}$. Taking advantage of the fact that $e^{\lambda t} \neq 0$ we can use algebra,

$$\frac{\mathbf{p}\lambda e^{\lambda t}}{e^{\lambda t}} = \frac{A\mathbf{p}e^{\lambda t}}{e^{\lambda t}}$$
$$A\mathbf{p} = \lambda\mathbf{p}$$

to find $A\mathbf{p} = \lambda\mathbf{p}$. Looking at the equation in this form we discover λ to be eigenvalues of A and $\mathbf{p}$ to be the corresponding eigenvectors. The stability at a critical point is thus determined by the sign of the real value of λ. If $Re(\lambda) > 0$ then $\mathbf{P}(t) = \mathbf{p}e^{\lambda t}$ goes to $\pm\infty$ and the critical point is unstable. If $Re(\lambda) \leq 0$ then $\mathbf{P}(t) = \mathbf{p}e^{\lambda t}$ is either oscillatory ($Re(\lambda) = 0$), or goes to 0 and the critical point is stable.

To determine the values of the eigenvalues we must look at the characteristic equation for A, i.e., $det(A - \lambda I_3) = 0$:

$$\begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - z_0 & -1 - \lambda & -x_0 \\ y_0 & x_0 & -\beta - \lambda \end{vmatrix} =$$
$$(\sigma + \lambda)[(1 + \lambda)(\beta + \lambda) + x_0^2] - \sigma[-(\rho - z_0)(\beta + \lambda) + x_0 y_0] = 0.$$

We then plug in the critical points and solve for λ. For example, consider $(x_0, y_0, z_0) = (0, 0, 0)$. Then:

$$(\sigma + \lambda)(1 + \lambda)(\beta + \lambda) + \sigma\rho(\beta + \lambda) = 0$$

$\iff$

$$(\beta + \lambda)[(\sigma + \lambda)(1 + \lambda) + \sigma\rho] = 0$$

$\iff$

$$(\lambda + \beta)[\lambda^2 + (1 + \sigma)\lambda + \sigma(\rho + 1)] = 0$$

From this we find solution is $\lambda = -\beta$. We can then use the quadratic formula to solve for the other values for λ:

$$\lambda = \frac{-(1+\sigma) \pm \sqrt{(1+\sigma)^2 - 4(1)\sigma(r+1)}}{2(1)}$$

λ can then be written in the form $\lambda = \lambda_r + i\lambda_{im}$ where λ_r is the real portion and λ_{im} is the imaginary portion. If the λ_r is positive then $x(t) = ke^{\lambda t}$ is unstable, if $\lambda_r \leq 0$ then $x(t)$ is stable and oscillatory.

Notice that if $(1+\sigma)^2 < 4(1)\sigma(r+1)2(1)$, then the only part of complex λ that determines the stability of the point is $\frac{-(1+\sigma)}{2(1)}$ since that is the real part of λ.

4. Determining Numerical Solutions

The initial method used to determine the numerical solutions and produce the figures for this project was the ODE 45 method. This routine is a built-in routine in Matlab and uses a variable step size Runge-Kutta Method to solve the set of differential equations numerically. It acts much like a "Black Box", where information is put in and a solutions comes out of the code. Specifics about what happens in the box are unknown to the user. We now describe the 4th order Runge-Kutta method given a step size h.

The 4th order Runge-Kutta is used to solve the initial value problem for a system of ordinary differential equations. One is given:

$$\frac{dy}{dx} = f(x, y),\ \ y(0) = y_0$$

where one solve for $y(t)$. The 4th order Runge-Kutta method for this problem is given by:

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),\ \ i = 1, 2, 3, ...N$$

where

$$k_1 = f(x_i, y_i), \qquad k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h), \qquad k_4 = f(x_i + h, y_i + k_3 h)$$

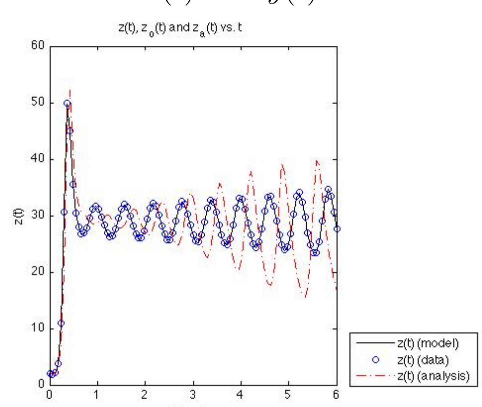
and h is our step size. In the ODE45 Matlab program, one cannot control the value of h. In an explicit code, one has absolute control over this value. We apply our 4th order Runge Kutta method to solving the Lorenz system of equations incorporating real data into the model. In order to do the data assimilation (i.e., data incorporation) technique correctly, we need explicit control of our step size h. We discuss this in the next section.

5. Data Assimilation

Data Assimilation uses observed past data and data from a prediction model, for us the Lorenz system, to produce analysis, which is considered the best estimate of the current state of the system. This is called the Analysis Step for sequential data assimilation. For the analysis step at time t we use the 4th order Runge-Kutta method to solve the Lorenz system for our model and use the method of successive corrections for our data analysis scheme. The method of successive corrections uses the equation:

$$y^a_{n+1} = y^b_n + K(H(y^b_n) - y^{obs}_n), \qquad n = 0, 1, 2, ...(N-1)$$

where y^a_n is our analysis solution, y^b_n is our background solution from the model, y^{obs}_n is the measured data at iterate n, $H = \frac{d\mathbf{P}(t)}{dt}$ is a correction factor and $K = \alpha I_3$ with α as a scalar which accounts for the variable change between observation and analysis. In the figure below, we use $N = 50$ iterates to solve our Lorenz system at each time step where physical data is incorporated into the solution at each time step. This is shown for only $z(t)$. Similar figures can be done for $x(t)$ and $y(t)$.



In the figure above of $z(t)$ the blue circles represent the measured data, the black line represents the solutions from the model, and the red dashed line represents the analysis.

6. Acknowledgements

In our future research, the aim is to find better value for K as to more accurately predict the data as well as look more deeply into data assimilation. I would like to think Dr. Horne for agreeing to work with me on this project and his continued guidance as I worked through it. I want to thank Dr. Peng for keeping me on task, Dr. Cooper for his assistance, my advisor Dr. Brania and the rest of the Morehouse College Mathematics Department as a whole, students and faculty alike.

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[2] Colin Sparrow. The Lorenz Equations: Bifurcations, Chaos, and Strong Attractors. Springer-Verlag New York Inc., 1982.

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Analysis of the Short Pulse Equation

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Abstract: This project analyzes the Schäfer-Wayne short pulse equation in the effort to obtain a traveling wave solution for light propagating through an optical fiber (see [**?**]). Normal light pulses propagate through an optical fiber with a pulse width in the order of picoseconds (ps). These pulses can be modeled using a traveling wave approximation to the Nonlinear Schrödinger equation. However, pulses propagating through an optical fiber with a pulse width in the order of femtoseconds (fs) do not have a traveling wave approximation using this equation. The Schäfer-Wayne short pulse equation is used instead, but is found not to have a traveling wave solution though it does have other mathematically sound solutions. The outcome of this research is to develop a method for modeling waves with ultra short pulse widths as they propagate through an optical fiber.

7. Introduction

Optical fiber technology has been proven useful in a variety of applications including communication, entertainment, and national defense. There are two types of fiber optical systems, fiber based and transmitter based. A typical fiber based optical system consists of a laser transmitter that propagates light through an optical fiber at a specific wavelength and varying power levels. Differences in power levels (known as burst duty factors) and wavelengths are purposely transmitted through these fibers to a receiver that notices differences in power levels and wavelengths and decodes these differences as either 1 or 0 to form a complete bit stream. A typical transmitter based optical system consists of a laser transmitter that propagates light through a short distance of optical fiber to an amplifier. The light is then amplified and sent to a transmitter where it is transmitted through the air to a receiver (usually very far away). The receiver recognizes key differences in power levels and wavelengths to form a bit stream that is used as information.

Optical Fibers themselves are glass fibers used to propagate pulses of light from a transmitter to a receiver. These fibers generally consist of a glass core with special cladding to prevent different effects environment can have on the propagating light. Most pulses sent through an optical fiber have a pulse width (τ) in the order of picoseconds(ps) because these pulses are generally easy to predict and thus are easy to implant information on. However, optical fibers can handle light propagation at an extremely wide range of frequencies. If there is a way to predict behavior of light at other frequencies, then these new frequencies can be used to send information at a high rate. The pulses of interest to be modeled in this project have pulsewidths in the order of $\tau \simeq 10 femtoseconds(fs)$

Currently the Nonlinear Schrödinger equation is used to predict light behavior propagating in $\tau \simeq$ picoseconds. This equation formed by physicist Erwin Schrödinger can be used for either light or water waves to model how the system that describes these waves changes over time. Solutions to this equation exist for a number of light waves and can be numerically analyzed if an analytical solution is not possible. The numerical methods of solving this equation for ultra-short pulse light behavior fail as a result of the differences in light behavior at these frequencies.

Authorities in ultra-short pulse light propagation T. Schafer and G.E. Wayne derived a partial differential equation from Maxwell's equation that describes light propagation at ultra-short pulse widths. An analytical or numerical solution to this equation could then be used to model light behavior at ultra-short pulse widths. If this model is effective enough, it can be applied to fiber optic systems.

This project will look at the Schafer-Wayne equation to model the propagation of pulses with an ultra-small pulse width. The equation, $u_{xt} = \alpha u + u^3_{xx}$ (where the subscripts represent partial derivatives; [1, 2]) lacks a traveling wave solution but features what is called the one loop solution. Since the one-loop solution is discontinuous, however, the Schafer-Wayne equation must be examined numerically. The One-Loop solution and numerical analysis will be implemented using Matlab and modified to add amplification effects.

8. Preliminaries

The specific Schäfer-Wayne short pulse equation in this project is given by:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

$$x, t \in \mathbb{R} \qquad \lim_{|x|\to\infty} u(x,t) = 0$$

$$\lim_{|x|\to\infty} \frac{\partial u}{\partial t}(x,t) = 0 \qquad \lim_{|x|\to\infty} \frac{\partial u}{\partial x}(x,t) = 0$$

where $u(x,t)$ represents a wave u traveling through a nonlinear medium at position x and time t, u_{xt} represents the pulse evolution, and $(u^3)_{xx}$ represents the nonlinear effects on the pulse as it propagates.

This can be rewritten as the partial differential equation of $f(\xi)$ where $\xi = (x + ct)$:

$$f''(\xi) = \frac{2[1 + (f'(\xi))^2]f(\xi)}{2c - f(\xi)^2}$$

$$\xi = (x + ct) \in \mathbb{R} \qquad \lim_{|\xi|\to\infty} f(\xi) = 0 \qquad \lim_{|\xi|\to\infty} f'(\xi) = 0$$

where $f(\xi)$ represents a wave traveling through a nonlinear medium.

9. Traveling Wave Solution to Short Pulse Wave Equation

We let $u(x,t) = f(x + ct) = f(\xi)$ be a traveling wave solution to the Schäfer-Wayne equation. This is done in a similar manner as applied to other nonlinear partial differential equations (see [3]). We now show, however, that traveling waves are not supported by this equation. We do, however, derive another type of solution. We start with the Schäfer-Wayne equation:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \tag{9.1}$$

We put $u(x,t) = f(x+ct)$ to determine u_x:

$$\begin{aligned}
Let\ u(x,t) &= f(\xi) \\
(where\ \xi = x+ct\ for\ c \in \mathbb{R}) & \\
\implies u_x &= \frac{\partial u}{\partial x} \\
\implies \frac{\partial u}{\partial x} &= \frac{df}{dx}\frac{\partial \xi}{\partial x} \\
\implies \frac{df}{dx}\frac{\partial \xi}{\partial x} &= f' \cdot 1 \\
\therefore u_x &= f'.
\end{aligned}$$

Now, we derive the u_{xt} term in terms of $f(x+ct)$:

$$\begin{aligned}
Let\ u_x &= f'(\xi) \\
(where\ \xi = x+ct\ for\ c \in \mathbb{R}) & \\
\implies u_{xt} &= \frac{\partial u_x}{\partial t} \\
\implies \frac{\partial u_x}{\partial t} &= \frac{\partial f'}{\partial t} \\
\implies \frac{\partial f'}{\partial t} &= \frac{df'}{d\xi}\frac{d\xi}{dt} \\
\implies \frac{df'}{d\xi}\frac{d\xi}{dt} &= c \cdot f'' \\
\therefore u_{xt} &= cf''.
\end{aligned}$$

Using the previously derived results above and plugging this into eq.(9.1), we find:

$$cf'' = f + \frac{1}{6}(f^3)_{xx}.$$

We examine the term $\frac{1}{6}(f^3)_{xx}$ to find:

$$\begin{aligned}
(f^3)_x &= 3f^2 \cdot f' \\
\implies (f^3)_{xx} &= \frac{d(f^3)_x}{dx} \\
\implies \frac{d(f^3)_x}{dx} &= 6f \cdot (f')^2 + 3f^2 \cdot (f'') \\
(by\ product\ rule) & \\
\therefore (f^3)_{xx} &= 6f(f')^2 + 3f^2(f'').
\end{aligned}$$

From this work, we obtain the following ordinary differential equation:

$$(f^2 - 2c)f'' + 2(1 + (f')^2))f = 0. \tag{9.2}$$

We multiply eq.(9.2) by the integrating factor:

$$2(f^2 - 2c)f'$$

to find:

$$\begin{aligned}
(f^2-2c)f''+2(1+(f')^2))f &= 0\\
\Longleftrightarrow 2(f^2-2c)^2f'f''+4(f^2-2c)(1+(f')^2)ff' &= 0\\
\Longleftrightarrow \frac{d}{d\xi}[(f^2-2c)^2(1+(f')^2) &= 0\\
\Longrightarrow (f^2-2c)^2(1+(f')^2) &= d_0\\
(\textit{where } d_0 \textit{ is some constant})
\end{aligned}$$

As $|\xi| \to +\infty$ we require $f, f' \to 0$. This requirement determines d_0.

$$\begin{aligned}
As\ |\xi| \to +\infty : (-2c)^2(1) &= d_0\\
4c^2 &= d_0
\end{aligned}$$

We now solve our nonlinear ordinary differential equation for f':

$$\begin{aligned}
(f^2-2c)^2(1+(f')^2) &= 4c^2\\
\Longleftrightarrow 1+(f')^2 &= \frac{4c^2}{(f^2-2c)^2}\\
\Longleftrightarrow (f')^2 &= \frac{4c^2}{(f^2-2c)^2}-1\\
\Longleftrightarrow \left(\frac{df}{d\xi}\right)^2 &= \frac{4c^2f^4+4f^2c-4c^2}{(f^2-2c)^2}\\
\Longleftrightarrow \left(\frac{df}{d\xi}\right)^2 &= \frac{f^2(4c-f^2)}{(f^2-2c)^2}\\
\Longrightarrow \left(\frac{df}{d\xi}\right) &= \pm\frac{f\sqrt{4c-f^2}}{(2c-f^2)}.
\end{aligned}$$

We note that the previous equation a singularity at $f=\pm\sqrt{2c}$ which implies there is no traveling wave solution to the equation $u_{xt}=u+\frac{1}{6}(u^3)_{xx}$. However, this does not mean there is no solution to the short-pulse wave equation.

10. One-Loop Solution to Short Pulse Wave Equation

We now derive the one-loop soliton solution associated with the SPE given by eq. (9.1). We begin with our ode:

$$\left(\frac{df}{d\xi}\right) = \pm\frac{f\sqrt{4c-f^2}}{(2c-f^2)}.$$

We solve this ode explicitly for $f(\xi) = f(x+ct)$ below:

$$\left(\frac{df}{d\xi}\right) = \pm\frac{f\sqrt{4c-f^2}}{(2c-f^2)} \iff \left(\frac{df}{d\xi}\right)\left(\frac{(2c-f^2)}{f\sqrt{4c-f^2}}\right) = \pm 1$$

$$\implies \int\left(\frac{(2c-f^2)}{f\sqrt{4c-f^2}}\right)\left(\frac{df}{d\xi}\right)d\xi = \pm\int d\xi \iff \int\left(\frac{(2c-f^2)}{f\sqrt{4c-f^2}}\right)df = \pm\xi$$

$$\iff \sqrt{4c-f^2} - \sqrt{c}\ sech^{-1}\left(\frac{f}{2\sqrt{c}}\right) = \pm[(x-x_0) = c(t-t_0)]$$

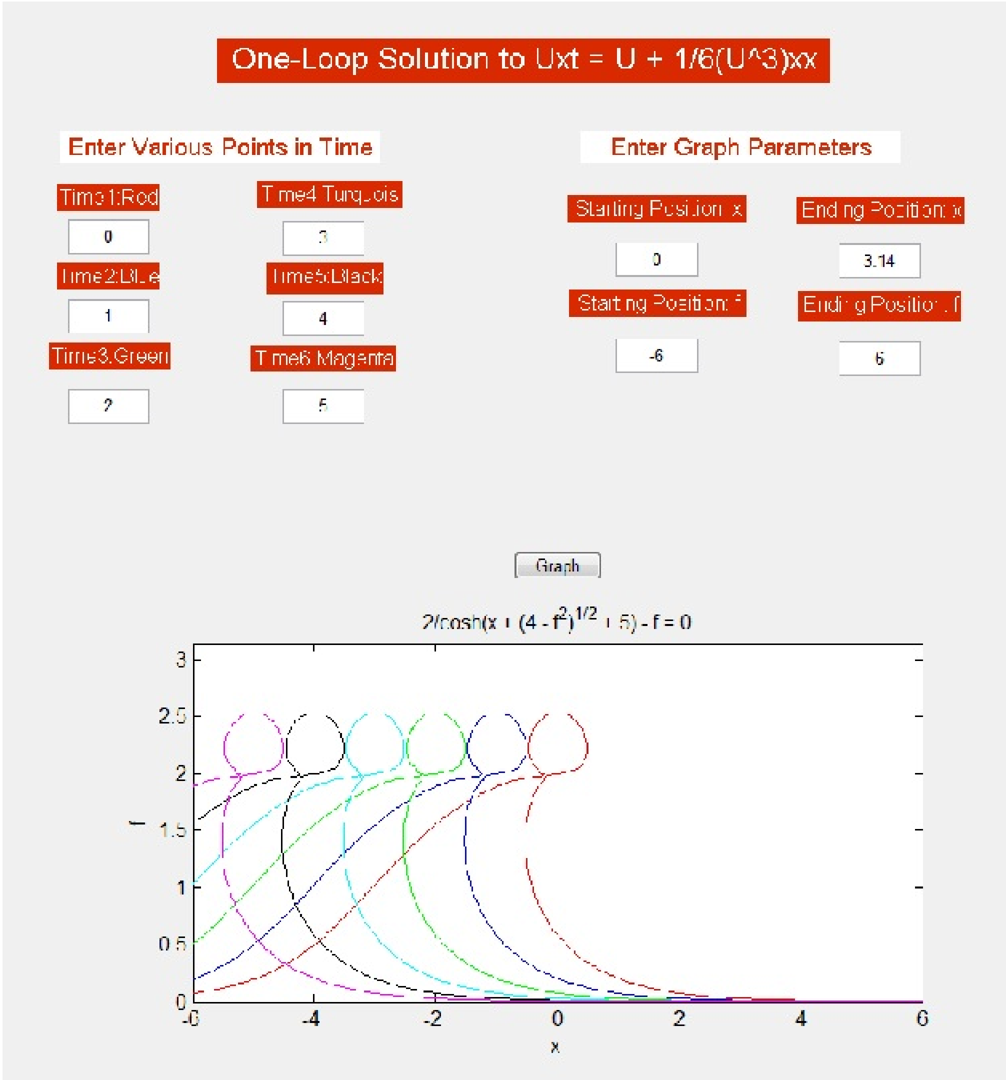
$$\iff f = 2\sqrt{c}\ sech\left[\frac{\sqrt{4c-f^2}}{\sqrt{c}} \mp \frac{x-x_0}{\sqrt{c}} \mp \sqrt{c}\ (t-t_0)\right].$$

For convenience we consider a special case of this equation where $c = 1$ and $y_0 = t_0 = 0$. We then have $f = 2sech[\sqrt{4-f^2} \mp (x+t)]$. If we set $y = \sqrt{4-f^2} \mp x$, then $x = \pm\sqrt{4-f^2} \mp y$ where $f = 2sech[y+t]$. This yields what is known as the one loop soliton solution. This solution appears as a single loop and "moves left" as various fixed values for t increase.

This solution was implemented using Matlab where that implementation was formed into a GUI in which a user can enter various values for t as well as enter various axis parameters. These parameters were then used to plot a graph of the one loop solution at these various times t along the given axis. A screenshot of this GUI is shown in Figure 4.1.

The above figure shows the actual Matlab GUI created to plot the one loop solution. Notice as the various times t increase the loop in the graph moves left.

Figure 4.1



11. Summary

Optical fiber systems are becoming a widely used method of data transmission. These systems are often limited in performance because the current method of modeling light behavior fails to approximate light behavior at extremely high frequencies (which correspond to ultra short pulse widths). If a method was formed to model light behavior at these frequencies were to be perfected it could be used to increase data rates in fiber optic systems and would also have security benefits associated with it. The Schăfer-Wayne short pulse equation was derived in 2004 in order to model such light behavior.

The goal of this project was to attempt a traveling wave solution to the Schăfer-Wayne short pulse equation analytically. Using normal methods of solving a partial differential equation, this equation was found not to have a traveling wave solution. By setting certain parameters, a mathematically sound solution (referred to as the One-Loop solution for its shape) was derived and implemented using Matlab. This Matlab implementation was then used to create a GUI in which a user could enter various values for time and set parameters for a axes of the plot. Once set, these parameters were used to plot the One-Loop solution using a Matlab plotting tool.

Future steps in this project involve using Matlab to implement a numerical approximation of the Schӓfer-Wayne short pulse equation. Once implemented, variables representing physical effects of light propagation are to be added to the Schӓfer-Wayne short pulse equation and the new equation will be approximated using the original Matlab implementation. Once this implementation is perfected, it will be used to make a user friendly GUI in which a user will be able to set parameters and variables and plot the approximation.

12. Acknowledgments

I would like to acknowledge my advisor Dr. Rudy Horne for guiding me through this research project; Dr. Chuang Peng, Dr. Duane Cooper, and the rest of the Mathematics Department at Morehouse College for this opportunity to produce this research project. I would also like to acknowledge my fellow classmates and peers.

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Biostatistics of Populations of Cancerous Cells

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Abstract:
The research is on Biostatistics of Cancerous Cells, using the probability and statistics in the realm of biology; Biostatistics. It covers the mean and variance of normal distributions and how these distributions are used. In addition to these distributions, the Odds and Risks Ratio, which are ratios mostly used in the scientific world, will be used in determining the biostatistics of cancer. It will address how cohort life tables are used to find the Odds and Risks Ratio of finding patients whom have cancer. These tables are also used to figure out if certain medicines are equipped for possible cures of cancer. Cohort life tables can be expressed as observed values or expected values; with the expected values being the sample mean or the estimates of the observed values. The analysis here will reflect the mathematical and biological process for the five stages that a medicine goes through in a scientific procedure. We are providing the connection between the biology and the mathematics.

13. **Introduction**

In general, cancer is responsible for millions if not billions of deaths every year. In the mist of lives being lost to this disease, doctors perform epidemiologic studies to test if there is a link or association between a variable such as smoking and an outcome, which in this case would be the increased or decreased risk of being infected with cancer. Most studies are created with a particular hypothesis. In examining the truth of a hypothesis, a null hypothesis is created; that the variable at hand is not associated with the risk of a disease. The null hypothesis is then tested using statistical methods and either accepted or rejected. The statistical methods that are used in the null hypothesis test are the Odds and Risks ratios along with the Z-test. The Odds ratio estimates chances of a particular event occurring in one population in relation to its rate of occurance in another population, while the Risk ratio compares the risk of developing a disease in people not receiving medical treatment vs. people who are receiving an established treatment. The Risk ratio is mostly used in evaluating clinical data while the Odds ratio is used not only in clinical data, but a variety of data. Both ratios are probability distributions with two random variables and are expressed using a cohort life table. These life tables are used to determine the likelihood of developing a disease. A more precise definition of cohort life tables; tables of data on survivorship of individuals within a population. In a clinical study, a cohort represents a well defined group of subjects or patients who have a common experience or exposure and then followed up for an incidence of new events. Each cohort table has information contained in $j \times j$ stratums. The data enclosed in each table are the individuals within a population who have survived a certain disease pending on the exposure to that disease. The results from the data can be compared from cohort to cohort which will enable an anaylsis of their variation. The other method, the Z-test is a statistical test for a distribution under the null hypothesis and can be approximated by a normal distribution; the most prominent probability used in statistics. Now, squaring the Z-test gives us what we call the chi-square test. The chi-square test is any statistical hypothesis test in which the distribution of the test is a chi-square distribution given that the null hypothesis is accepted. The chi-square test is used for measuring how close the total number of categorical variables (variables that place individuals into categories) are to the value we expect under the null hypothesis. Examples of these chi-square tests are as follows: Likelihood Ratio Test, Pearon's Test and Wald's Test.

The Chi-Square Distribution is special case of the Gamma Distribution which are both probability density functions (p.d.f.) that consists of the real constants: alpha and beta. These probability denstiy functions are probabilities that are continuous over a particular interal. In the case of the Chi-square distribution, the constants alpha and beta are equal to $\frac{n}{2}$ and 2, respectively. This term n is known as the degrees of freedom. The degree of freedom is the number of individual samples. It is also defined as the number of independent variables, in conjunction with, the sample mean to find the dependent variables. Each chi-square distribution is specified by given its degrees of freedom. In biology, this parameter n or the degrees of freedoms are the number of rows minus one, times the number of columns minus one. These rows and columns are in reference to the cohort life table that we have previously referred to. The relationship between the degrees of freeedom in mathematics and biology is that the mean and the variance of a chi-square distribution with ν degrees of freedom is equal to n and $2n$, respectively. The mean and the variance are summations of all the elments in a sample.

In this paper these concepts of Odds and Risk Ratio, Cohort life tables, the different Chi-square distributions and the degree of freedom will be brideged together to show the relevance of mathematics in clinical studies. More importantly this paper provides the link that proves that the degree of freedom used in evaluating cohort life tables is technically sound with support from the mathematical background.

14. Preliminaries

This here is the definition of the identity of a probability function.

$$\sum_x P(X = x) = 1$$

The sum of all probabilities is always equal to 1. The summation is over all the elements in the sample space of X. We will look at two dimensions of probability; the mean and the variance. The mean of X or the expected value of X is defined to be

$$E(X) = \sum_x x\, P(X = x)$$

The variance of X is be defined to be

$$var(X) = \sum_x [x - E(X)]^2\ P(X = x)$$

The mean and the variance are constants that are finite. They are not random variables and are unknown in most applications. When the random variable of a probability function is derived, then through an explicit formula a transformation can be used to determine new random variables from the existing ones. Random variables are variables which have values that are that are numerical outcomes and include factors such as age, weight, sex and country in epidemiology studies.

A probability function containing two or more random variables simultaneously is called a joint probability function (j.p.f.). The j.p.f. for a pair of random variables (X,Y) is denoted by P(X=x, Y=y). Assuming that the sample space of a joint probability function is a set of pairs (x,y), x is the sample space of X, and Y is the sample space of y. So, similarly to the identity of a probability function,

$$\sum_x \sum_y P(X = x \ , Y = y) \ = \ 1$$

The random variables of X and Y are considered to be a unit. Therefore, from a joint probability function we can obtain the marginal probability functions, (m.p.f.) through the separation of the distribution of X. We can accomplish this by "summing over" Y to obtain the m.p.f. of X, and vice versa to obtain the m.p.f. for Y;

$$P(X = x) \ = \ \sum_y P(X = x \ , Y = y), \quad P(Y = y) \ = \ \sum_x P(X = x \ , Y = y)$$

If X,Y or X_1, X_2 are independent variables and the joint probability function is the product of the marginal probability function then,

$$P(X = x \ , Y = y) \ = \ P(X = x) \ P(Y = y)$$

In the case of the two independent variables X_1 and X_2, the mean is implied to be

$$E(X_1 + X_2) \ = \ E(X_1) + E(X_2), \quad E(X_1 - X_2) \ = \ E(X_1) - E(X_2)$$

With the variance being,

$$var(X_1 + X_2) \ = \ var(X_1 - X_2) \ = \ var(X_1) + var(X_2)$$

Given that X_1, X_2,....., X_n are independent and have the same distribution, X_i is the sample from the distribution and n is the sample size. In this assignment we are assuming that all of the samples are simple random samples and that the distribution is unspecified. Hence, we denote the mean of X_i by μ and the variance of X_i by σ^2. So, the sample mean is defined to be

$$\bar{X} \ = \ \frac{1}{n} \sum_{i=1} X_i$$

where $E(\bar{X}) \ = \ \mu$ and $var(\bar{X}) \ = \ \frac{\sigma^2}{n}$.

The standard deviation is the positive square root of variance and denoted by σ.

From the identity of a probability function, if we consider the two constants a and b then we will have $aX + b$. Using the fact that $g(x) = (aX + b)$

Proof:

$$E(aX + b) \ = \ aE(X) + b = \int_{-\infty}^{\infty} (ax + b) \ f(x) \, \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} ax \ f(x) \, \mathrm{d}x \ + \int_{-\infty}^{\infty} b \ f(x) \, \mathrm{d}x = a \int_{-\infty}^{\infty} x \ f(x) \, \mathrm{d}x \ + b \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = aE(X) + b$$

By setting $a = 0$ or $b = 0$ we have the following corollaries:

Corollary 1, a is constant, $E(aX) = aE(X)$.

Corollary 2, b is constant, $E(b) = b$.

The variance that we have defined earlier is also equal to:

$$E(X - \mu)^2 = E(X^2) - 2\mu E(X) + E(\mu^2) = E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - 2\mu^2 + \mu^2$$
$$= E(X^2) - \mu^2 = \mu_2' - \mu^2 = \sigma^2$$

Similarly, if we add two constants a and b then we will again have $aX + b$. Using the information stated above, the variance for $(aX + b)$ will be

$$var(aX + b) = \int_{-\infty}^{\infty} (ax + b)^2 \ f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} a^2x^2 + 2abx + b^2 \ f(x)\,\mathrm{d}x$$

$$= a\int_{-\infty}^{\infty} x^2 \ f(x)\,\mathrm{d}x + 2ab\int_{-\infty}^{\infty} x \ f(x)\,\mathrm{d}x + b^2\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = a^2\mu_2' + 2ab\mu_1 + b^2$$

If we set $b = 0$ and knowing that $\mu_2' = \sigma^2$ then,

$$var(aX + b) = a^2\sigma^2$$

Since we know that $E(aX+b) = aE(x)+b$ we can use this equation to solve other equations. Given $a = \frac{1}{\sigma}$ and $b = \frac{-\mu}{\sigma}$ we can find that the solution to

$$E(\frac{X-\mu}{\sigma}) = \frac{E(X)}{\sigma} - \frac{\mu}{\sigma} = 0$$

and

$$var(\frac{X-\mu}{\sigma}) = var[X\frac{1}{\sigma} - (\frac{-\mu}{\sigma})] = var(\frac{x}{\sigma} + \frac{\mu}{\sigma}) = (\frac{1}{\sigma})^2 \ var(X) = \frac{1}{\sigma^2} \ var(X) = 1$$

Another way of describing a probability distribution or function is by using a moment generating function. The moment generating function of a random variable X is:

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} \ f(x)$$

when X is discrete

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \ f(x) \ \mathrm{d}x$$

when X is continuous. The moment will be a sum when it is discrete and an integral when it is continuous. The moments are also differentiable at zero. The mean and varinace of the moments are as follows given $t = 0$:

$$M(t) = E(e^{tX}), \qquad M'(t) = E(Xe^{tX}), \quad M'(0) = E(X) = \mu$$

Since the variance is $E(X-\mu)^2 = E(X^2) - \mu^2$, $M(t) = E(e^{tX})$,

$$M'(t) = E(Xe^{tX}) = E(X^2e^{tX})$$

$$M''(0) = E(X^2), \quad [M'(0)]^2 = \mu^2, \quad M''(0) - [M'(0)]^2 = \sigma^2$$

The Gamma function is an integral,

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\,\mathrm{d}t$$

For any positive interger (n) the function is a factorial, $\Gamma(n) = (n-1)!$. In the special case of the Gamma of $\frac{1}{2}$, which it's significance we shall see later, the proof is laid down before us:

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t, \quad \Gamma(\frac{1}{2}) = \int_0^{\infty} t^{\frac{-1}{2}}e^{-t}\,\mathrm{d}t$$

let $u = t^{\frac{1}{2}}$, $\mathrm{d}u = \frac{1}{2}t^{\frac{-1}{2}}\,\mathrm{d}t$, $2\mathrm{d}u = t^{\frac{-1}{2}\mathrm{d}t}$, $2\int_0^{\infty} e^{-u^2}\mathrm{d}u$, $\mathrm{I} = \int_0^{\infty} e^{-u^2}\mathrm{d}u$, $\mathrm{I}^2 = \int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\,\mathrm{d}y\mathrm{d}x$,

$$\mathrm{I}^2 = \int_0^{\frac{\pi}{2}}\int_0^{\infty} re^{-r^2}\,\mathrm{d}r\mathrm{d}\theta$$

let $\mathrm{u} = r^2$, $\mathrm{d}u = 2r\mathrm{d}r$, $\frac{1}{2}\mathrm{d}u = r\,\mathrm{d}r$,

$$\frac{1}{2}\int_0^\infty e^{-u}\mathrm{d}u = \frac{-1}{2}e^{-r^2}, \quad \mathrm{I}^2 = \int_0^{\frac{\pi}{2}} \frac{-1}{2}e^{-r^2}\ \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \frac{-1}{2}(0-1)\mathrm{d}\theta = \frac{1}{2}\theta\ \frac{\pi}{2} = \frac{\pi}{4}$$

$$\mathrm{I} = \frac{\sqrt{\pi}}{2}, \qquad 2\int_0^\infty e^{-u^2}\mathrm{d}u = \sqrt{\pi}, \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Gamma satifies the following property,

$$\int_0^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}\ \mathrm{d}x = 1$$

where α and β are real constants that are always greater than zero. These two parameters are a stable for Gamma and also seen in its distirbution. The Gamma distribution is a family of continuous distributions which has the random variable X with the density function

$$f(x) = kx^{\alpha-1}\,e^{-\frac{x}{\beta}}$$

where the constant

$$k = \frac{1}{\beta^\alpha\Gamma(\alpha)}$$

given us the Gamma Distribution

$$\Gamma(t) = \int_0^t \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha\Gamma(\alpha)}\ \mathrm{d}x.$$

Given the property of Gamma, we can calculate the moment, mean and variance of the Gamma Distribution. The Moment Generating Function for Gamma is

$$M_x(t) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty e^{xt}\,x^{\alpha-1}\,e^{\frac{-x}{\beta}}\ \mathrm{d}x, \quad k = \frac{1}{\beta^\alpha\Gamma(\alpha)}, \quad M_x(t) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}\,e^{-x(\frac{1}{\beta}-t)}\ \mathrm{d}x.$$

Let $y = x(\frac{1}{\beta} - t)$.

$$M_x(t) = \frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \left(\frac{y}{\frac{1}{\beta}-t}\right)^{\alpha-1} e^{-y}\,\frac{1}{(\frac{1}{\beta}-t)}\ \mathrm{d}y = \frac{1}{\beta^\alpha\Gamma(\alpha)}\left(\frac{1}{\frac{1}{\beta}-t}\right)^\alpha \int_0^\infty y^{\alpha-1}e^{-y}\ \mathrm{d}y$$

Since, $\int_0^\infty y^{\alpha-1}e^{-y}\ \mathrm{d}y = \Gamma(\alpha)$, Therefore,

$$M_x(t) = \frac{1}{\beta^\alpha}\left(\frac{1}{\frac{1}{\beta}-t}\right)^\alpha = \left(\frac{1}{1-\beta t}\right)^\alpha = (1-\beta t)^\alpha = (1-\beta t)^{-\alpha}$$

The mean of the Gamma Distribution is

$$E(X) = \int_0^\infty x\frac{x^{\alpha-1}}{\beta^\alpha\Gamma(\alpha)}e^{\frac{-x}{\beta}}\ \mathrm{d}x = \int_0^\infty \frac{x^{\alpha+1}-1}{e}^{\frac{-x}{\beta}}\ \beta^\alpha\Gamma(\alpha)\,\mathrm{d}x$$

$$= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty \frac{x^{\alpha+1-1}}{e}^{\frac{-x}{\beta}}\ \beta^{\alpha+1}\Gamma(\alpha+1) = \frac{\beta(\alpha+1-1)\Gamma(\alpha+1-1)}{\Gamma(\alpha)} = \alpha\beta$$

The variance of the Gamma Distribution is

$$E(X^2) = \int_0^\infty \frac{x^{\alpha+2-1}}{\beta^\alpha\Gamma(\alpha)}\,e^{-\frac{x}{\beta}}\ \mathrm{d}x = \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha\Gamma(\alpha)}\ \int_0^\infty x^{\alpha+2-1}e^{\frac{-x}{\beta}}\beta^{\alpha+2}\Gamma(\alpha+2)\,\mathrm{d}x$$

$$= \beta^2(\alpha+1)(\alpha)$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$

Now the Beta function, which we will speak on breifly, is defined by the density

$$f(x) = kx^{\alpha-1}(1-x)^{\beta-1} \text{ where } 0 \le x \le 1 \text{ and } k = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) * \Gamma(\beta)}$$

where $\int_0^1 f(x)\,\mathrm{d}x = 1$.

Since, $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)*\Gamma(\beta)}$ is a constant, we know that the integral is equal to the reciprical of the constant:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha) * \Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Beta is expressed in terms of Gamma and holds the property,

$$B(\alpha,\beta) = \frac{\Gamma(\alpha) * \Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and therefore a clear relationship with Gamma is stated. The Beta, Gamma connection is as follows: Given two random variables that are incomplete gamma functions, $\Gamma(a_1, 1)$ and $\Gamma(a_2, 1)$, where the definition of an incomplete random variable is defined as

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t$$

a function in which the integrating factor (x) does not begin from zero, but from another real positve number if X_1 is a $\Gamma(a_1, 1)$ random variable and X_2 is a $\Gamma(a_2, 1)$ random variable then $\frac{X_1}{X_1+X_2}$ is a $B(a_1, a_2)$ a Beta random varible. Given the property of Beta and its connection with Gamma we are able to find the mean and variance of the Beta Distribution.

The mean of the Beta Distribution is

$$E(X) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x\, x^{\alpha+1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{x^{\alpha+1-1}(1-x)^{\beta-1}}{\Gamma(\alpha+1+\beta)\,\mathrm{d}x}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \int_0^1 \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha+1-1}(1-x)^{\beta-1}\,\mathrm{d}x$$

This integral equals 1. Therefore, $E(X) = \frac{\alpha}{\alpha+\beta}$.

The variance of the Beta Distribution is

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma\beta} x^{\alpha+2-1} e^{\frac{-x}{\beta}}\,\mathrm{d}x$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma\beta}{\Gamma(\alpha+2+\beta)} \int_0^1 \frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma\beta} x^{\alpha+2-1} e^{\frac{-x}{\beta}}\,\mathrm{d}x$$

This integral equals 1 since,

$$\Gamma(\alpha+2) = (\alpha+2-1)\Gamma(\alpha+2-1)$$

$$= (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)(\alpha+1-1)\Gamma(\alpha+1-1) = (\alpha+1)\,\alpha\,\Gamma(\alpha)$$

This integral equals 1. Therefore,

$$E(X^2) = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\sigma^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - (\frac{\alpha}{\alpha+\beta})^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

The Chi-square Distribution (χ^2) is a special case of the gamma distribution where the constant $\alpha = \frac{n}{2}$ and the constant $\beta = 2$. The mean and variance of the Chi-square Distrbution, with density function f(x) are n and $2n$ respectively,

$$f(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n-2}{2}} e^{\frac{-x}{2}} \text{ for } x \geq 0$$

We will prove that the mean and variance of the Chi-square distribution using the density function above and by using the Z-test respectively.

PROOF I. Mean of Chi-Square Distribution using Gamma Distribution,

$$E(X) = \int_0^\infty x \frac{x^{\alpha-1}}{\beta^\alpha\Gamma(\alpha)} e^{\frac{-x}{\beta}} \, \mathrm{d}x = \int_0^\infty x^{\alpha+1-1} e^{\frac{-x}{\beta}} \beta^\alpha \Gamma(\alpha) \, \mathrm{d}x$$

$$= \frac{\beta^{\alpha+1}\Gamma(\alpha+1)}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty x^{\alpha+1-1} e^{\frac{-x}{\beta}} \beta^{\alpha+1}\Gamma(\alpha+1) = \frac{\beta(\alpha+1-1)\Gamma(\alpha+1-1)}{\Gamma(\alpha)} = \alpha\beta$$

Plugging in $\alpha = \frac{n}{2}$ and $\beta = 2$, $E(X) = n$.

Variance of the Chi-Square Distribution using Gamma Distribution

$$E(X^2) = \int_0^\infty \frac{x^{\alpha+2-1}}{\beta^\alpha\Gamma(\alpha)} e^{\frac{-x}{\beta}} \, \mathrm{d}x = \frac{\beta^{\alpha+2}\Gamma(\alpha+2)}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty x^{\alpha+2-1} e^{\frac{-x}{\beta}} \beta^{\alpha+2}\Gamma(\alpha+2) \, \mathrm{d}x = \beta^2(\alpha+1)(\alpha)$$

$$\sigma^2 = E(X^2) - (\alpha\beta)^2 = \alpha\beta^2$$

Plugging in $\alpha = \frac{n}{2}$ and $\beta = 2$, $\sigma^2 = 2n$.

PROOF II Mean and Variance of Chi-Squared using Z-test. Since the chi-squared distribution has an additive property like the normal distribution and

$$Z = \frac{X-\mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow \frac{(X-\mu)^2}{\frac{\sigma^2}{n}} = Z^2$$

being that Z is standard normal $N(0,1)$, then Z^2 the transformation of Z is obtained by squaring Z which is chi-squared. Now, where Z is the random variable with standard normal distribution $N(0,1)$. Now for Z, $E(Z) = 0$:

$$E(Z^2) = E[(Z-0)^2] = E[(Z-\mu_z)^2] = var(Z) = 1.$$

Finding var(Z^2):

$$\text{var}(Z^2) = E[(Z^2 - \mu_{Z^2})^2] = E[(Z^2-1)^2] = E(Z^4 - 2Z + 1) = E(Z^4) - 2E(Z^2) + 1E(Z^4) - 2\times1+1$$
$$= E(Z^4) - 1$$

To find $E(Z^4)$, we will use the fact that for any continuous random variable X with p.d.f. f, and any exponent k,

$$E(X^k) = \int_{-\infty}^\infty x^k f(x) \, \mathrm{d}x$$

and that the p.d.f. f of the $N(0,1)$ random variable is given by $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$. Hence, using integration by parts,

$$E(Z^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty z^4 e^{\frac{-z^2}{2}} \, \mathrm{d}z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty z^3 z e^{\frac{-z^2}{2}} \, \mathrm{d}z = [z^3(-e^{\frac{-z^2}{2}})] - \int_{-\infty}^\infty 3z^2(-e^{\frac{-z^2}{2}}) \, \mathrm{d}z$$

$$= \frac{1}{\sqrt{2\pi}}[0 + \int_{-\infty}^{\infty} 3z^2 e^{\frac{-z^2}{2}} \,\mathrm{d}z = 3 \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{\frac{-z^2}{2}} \,\mathrm{d}z = 3E(Z^2) = 3 \times 1 = 3$$

Hence,

$$var(Z^2) = E(Z^4) - 1 = 3 - 1 = 2.$$

Therefore, an alternative way to define the chi-square distribution is by assuming that $Z_1, Z_2, ..., Z_n$ are independent random variables having standard normal distributions $N(0,1)$. Then the chi-square distribution, χ^2 is defined by

$$\chi^2 = {Z_1}^2 + {Z_2}^2 + ... + {Z_n}^2$$

Taking the expected value and variance of both sides,

$$E({\chi_n}^2) = E({Z_1}^2 + {Z_2}^2 + ... + {Z_n}^2) = E({Z_1}^2) + ... + E({Z_n}^2) \qquad \text{and}$$

$$\text{var } ({\chi_n}^2) = \text{ var } ({Z_1}^2 + {Z_2}^2 + ... + {Z_n}^2) = \text{ var } ({Z_1}^2) + ... + \text{ var } ({Z_n}^2)$$

But all the instances of Z_i have identical distributions, so

$$E({\chi_n}^2) = \text{ n } E(Z^2) = n(1) = n \quad \text{ and } \quad \text{var}({\chi_n}^2) = \text{ n } \text{var}(Z^2) = 2n$$

Now, we see that for any n the two defintions of Chi-Square are the same. Here, we will prove the case where $n = 1$. We will use the Derivation for one degree of freedom.

PROOF. Let random variable Y be defined as $Y = Z^2$ where Z has the normal distribution with mean 0 and variance 1 ($N(0,1)$). Then if $y < 0$, $P(Y < y) = 0$ and if $y \geq 0$,

$$P(Y < y) = P(Z^2 < Y) = P(|Z| < \sqrt{y})$$

$$= F_Z(\sqrt{y}) - F_Z(-\sqrt{y}) = F_Z(\sqrt{y}) - (1 - F_X(\sqrt{y}) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{2\partial F_Z(\sqrt{y})}{\partial y} - 0 = 2(\int_{-\infty}^{\sqrt{\pi}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} \,\mathrm{d}t)' = 2\frac{1}{\sqrt{2\pi}}\, e^{\frac{-y}{2}} (\sqrt{y})'$$

$$= 2\frac{1}{\sqrt{2}\sqrt{\pi}}\, e^{\frac{-y}{2}} (\frac{1}{2}\, y^{\frac{-1}{2}}) = \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} y^{\frac{1}{2}-1}\, e^{\frac{-y}{2}}$$

where f is the p.d.f. of the corresponding random varibales. Then, $Y = Z^2 = {\chi_n}^2$, which is the same result we have from the proof of taking the square of the summation of the corresponding random variables. The derivation of the p.d.f. for two degrees of freedom follows a similar pattern, but we will not go into detail at this time. Using this derviation, we see how the $\Gamma(\frac{1}{2})$ plays a key role in connecting the two ways that we define the chi-square distribution.

The parameter n is called the degrees of freedom. The degrees of freedom (df) is the number of independent variables. The Chi-Square distribution are determined by the degrees of freedom. Therefore, there is a different distribution for each degree of freedom. In the case of the sample mean, $\bar{X}$, the degrees of freedom is (n-1). The sample mean, discussed earlier in the preliminaries, is used when using a sample population from an even larger population. The (df) of a sample population is n-1 beacuse there are the (n-1) variables that we can freely choose to be any value. The other value, n, has to be fixed in order for the sample mean to maintain its original value. This n is dependent on the other (n-1) independent values. Hence, the term degrees of freedom.

Chi-square will help us determine if a study is large enough and statistically significant after observing the Odds and Risk Ratios. The odds ratio estimates chances of a particular event ocurring in one population in relation to its rate of occurrence in another population.

In mathematical terms the odds ratio is the joint probability distribution of two binary variables.

	$Y=1$	$Y=0$
$X=1$	a	b
$X=0$	c	d

where a,b,c and d are non-negative cell probabilities that sum to 1. The odds for Y within the two sub-populations of X are defined in terms of the conditional probabilities given X:

	$Y=1$	$Y=0$
$X=1$	$\frac{a}{(a+b)}$	$\frac{b}{(a+b)}$
$X=0$	$\frac{c}{(c+d)}$	$\frac{d}{(c+d)}$

Giving us the odds ratio,

$$OR = \frac{ad}{bc}$$

The risk ratio is the risk of an event (or of developing a disease) relative to exposure. Relative risk is a ratio of the probability of the event occurring in the exposed group vs. a non exposed group.

$Risk$	$Disease\, status(present)$	$Disease\, status(absent)$
$Smoker$	a	b
$Non-smoker$	c	d

Here, we can determine the risk ratio to be,

$$RR = \frac{P_{exposed}}{P_{nonexposed}} \quad RR = \frac{\frac{a}{(a+b)}}{\frac{c}{(c+d)}}$$

A relative risk of 1 means there is no difference in risk between the two groups. An RR of <1 means the event is less likely to occur in the experimental group than in the control group. An RR of >1 means the event is more likely to occur in the experimental group than in the control group.

Moreover, the relationship between the odds and risk ratio is that if a is is substantially smaller than b then,

$$\frac{a}{a+b} \approx \frac{a}{b}$$

and when c is substantially smaller than d then,

$$\frac{c}{c+d} \approx \frac{c}{d}$$

Thus,

$$RR = \frac{\frac{a}{(a+b)}}{\frac{c}{(c+d)}} \approx \frac{ad}{bc} = OR$$

Now, the tables that we have used for the Odds and Risk ratios are not picked randomly. They are called cohort life tables. Cohort life tables are tables of data on survivorship within a population. These tables can be used to determine age, survivorship, the likelihood to develop a disease and many other things. A cohort is a group or a band of people and their tables are used to determine the likelihood of developing a disease by observing the Odds and Risk Ratios. Now, the Observed Counts of a Closed Cohort Study is represented by the cohort:

a	b	m_1
c	d	m_2
r_1	r_2	r

And the Expected Counts of a Closed Cohort study is represented by the cohort:

$\hat{e}$	$\hat{f}$	m_1
$\hat{g}$	$\hat{h}$	m_2
r_1	r_2	r

where the rows represent if a population has a disease or not and the columns represent if a population was exposed or not exposed to a factor that can potentially cause a disease. Now, $m_1 = a + b$, $m_2 = c + d$, $r_1 = c + d$, $r_2 = b + d$, and $r = a + b + c + d$. Now the expected value is

$\hat{a}$	$\hat{b}$	m_1
$\hat{c}$	$\hat{d}$	m_2
r_1	r_2	r

where $\hat{a} = \frac{r_1 m_1}{r}$, $\hat{b} = \frac{r_2 m_1}{r}$, $\hat{c} = \frac{r_1 m_2}{r}$, $\hat{d} = \frac{r_2 m_2}{r}$. Every random variable that contains a hat $\hat{a}$ is an estimation of the observed counts of a closed cohort study. In all cases $\hat{a} = \hat{e}$, $\hat{b} = \hat{f}$, $\hat{c} = \hat{g}$, $\hat{d} = \hat{h}$. With where π and ω are probabilities; we use $\omega = \frac{\pi}{1-\pi}$ and solving for π we come up with $\pi = \frac{\omega}{1+\omega}$. We then substitute

$$\frac{m_1}{r} = \pi_1 \quad ; \quad \frac{m_2}{r} = \pi_2$$

Thus, we see that

$$\hat{a} = \pi_1 r_1, \quad \hat{b} = \pi_1 r_2, \quad \hat{c} = \pi_2 r_1, \quad \hat{d} = \pi_2 r_2$$

In terms of ω and π we conclude that the odds ratio or the $OR = \frac{\omega_1}{\omega_2} = \frac{\pi_1(1-\pi_2)}{\pi_2(1-\pi_1)}$, where π and ω are probabilities Using the information that we know about ω_1 and ω_2 we see again that $OR = \frac{ad}{bc}$.

$$\hat{\omega_1} = \frac{\hat{\pi_1}}{1-\hat{\pi_1}} = \frac{a}{r_1 - a} = \frac{a}{c}, \quad \hat{\omega_2} = \frac{\hat{\pi_2}}{1-\hat{\pi_2}} = \frac{b}{r_2 - b} = \frac{b}{d}$$

Since, $a + c = r_1$; $b + d = r_2$; $\hat{\pi_1} = \frac{a}{r_1}$; $\hat{\pi_2} = \frac{b}{r_2}$. In this case, using the cohort life tables as before we find that again; the risk ratio or $RR = \frac{\pi_1}{\pi_2} = \frac{m_1}{m_2} = \frac{ar_2}{br_1}$.

In order to obtain chi-square, we will use the Z-test,

$$Z = \frac{X-\mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow \frac{(X-\mu)^2}{\frac{\sigma^2}{n}}$$

a test from which a distribution under the null hypothesis can be approximated by a normal distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{(-\frac{1}{2})x^2}$$

Under the null hypothesis $H_0 : \pi = \pi_0$ where π_0 is a given value. The estimates of the mean and variance of $\hat{\pi}$ are $E_0(\hat{\pi}) = \pi_0$ and $var_0(\hat{\pi}) = \frac{\pi_0(1-\pi_0)}{r}$.

A test of the null hypothesis is

$$\chi^2 = \frac{r(\hat{\pi} - \pi_0)^2}{\pi_0(1-\pi_0)} = \frac{(x - r\pi_0)^2}{\pi_0(1-\pi_0)r} \quad (df = 1)$$

where $x = r\pi$. χ^2 is equal to the mean of the random variable squared divided by the variance of the random variable.

Another way to express χ^2 is the sum of the observed minus the expected squared divided by the expected. The *df* is the degrees of freedom is equal to the number of rows(m) minus one (m-1) times the number of columns(p) minus one (p-1) using the cohort life tables.

$$Df = (m-1) * (p-1)$$

This is due to the fact that you can manipulate or freely choose the population of a those with a disease (rows) and the population of those who are not exposed to a factor contributing to the disease.

15. Summaries

The project has given insight to the relationship between the mathematics and the biology realm; more precisely how statistics is used in biological studies of disease. Most biologist do not understand where the formulas for the chi-square distribution and the degree of freedom originate from, nor how their names came about. Now we know that the term chi-square is derived from Z-test

$$Z = \frac{X - \mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow \frac{(X-\mu)^2}{\frac{\sigma^2}{n}} = Z^2$$

and by squaring the Z-test or the independent variables of a sample with Z being standard normal,

$$\chi^2 = {Z_1}^2 + {Z_2}^2 + ... + {Z_n}^2$$

and that the formula for the chi-square distribution given by the density,

$$f(x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n-2}{2}} e^{\frac{-x}{2}} \text{ for } x \geq 0, \quad \text{is} \quad \int_0^x f(x)\,\mathrm{d}x = \int_0^x \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} x^{\frac{n-2}{2}} e^{\frac{-x}{2}}\,\mathrm{d}x$$

which comes from the gamma distribution where the real constants alpha and beta are $\frac{n}{2}$ and 2.

The Z-test gives way to the null hypothesis test of chi-square, which scientists use in experiments to determine whether their data is statistically significant and if they should accept or reject a given piece of data.

The term n, known as the degree of freedom is a parameter in the chi-square distrbution which gives us the number of independent samples. In biology, the degree of freedom is used in sample populations to manipulate certain factors of a specific population. We now know that the degree of freedom in sample populations are n-1 beacuse these are the number of variables that we can freely choose to be any value. The last value is dependent on the other values and has to be fixed in order to be added with the other values to equal the sample population.

16. Acknowledgements

I would like to thank my classmates, my professor and advisor Dr. Peng, the chairman of our math department Dr. Cooper and the entire mathematics department for their guidance throughout project.

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Hermite Interpolation Interpolation of Perturbed Elliptic Functions and Their Integrals

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Abstract:
In this paper, we derive Hermite polynomial approximation of perturbed elliptic functions and subsequently evaluate their integrals on closed intervals of the real line. Continuity of such elliptic functions is assumed but obtaining closed form integrals solutions of such integrals are challenging, hence, adoption of numerical integration techniques will be in order. The results are compared with Taylor Polynomials and Maclaurin Polynomials as well as the binomial series of the elliptic function. Examples are presented for illustration.

17. Introduction

In many fields of science and engineering, evaluating integrals of certain class of functions could become a very difficult task especially if the function which is to be evaluated cannot be evaluated in closed form. Consider the evaluation of the integral:

$$I = \int_a^b g(x)dx \tag{17.1}$$

where $g(x)$ is a continuous function in the interval $[a, b]$. In our case under study, it could be impracticable to evaluate the integral in closed form, hence a method of approximating $g(x)$ by an elementary function is in order. Since polynomials are continuous functions in every interval of the real line, approximating $g(x)$ by a polynomial is admissible. By Weierstrauss Theorem in Ref. [1] (Burden and Faires), there exists a polynomial $P_n(x)$ of degree n such that $|g(x) - P_n(x)| \leq \epsilon$ for every point in the interval $[a, b]$, where ϵ is a very small positive number ($\epsilon = O(10^{-6})$). We shall use various types of polynomial interpolation to approximate an elliptic function of the form:

$$h(y) = \frac{1}{\sqrt{\frac{1}{3} - k^2\sin^2(y) + \frac{1}{15}y^2}}, \qquad 0 < k < \frac{1}{\sqrt{3}} \tag{17.2}$$

and subsequently approximate

$$\Psi = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{1}{3} - k^2\sin^2(y) + \frac{1}{15}y^2}}dy. \tag{17.3}$$

We shall examine results generated using the Taylor polynomial approximation, Lagrange Interpolating polynomial approximation and extend this work to Hermite polynomial approximations. A polynomial of the form:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{17.4}$$

or put in the form:

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} \tag{17.5}$$

is called the n^{th} Taylor polynomial centered at x_0. Taylor polynomials can be used to approximate continuously differentiable functions in a neighborhood of a point (our point is x_0). Now, if the center $x_0 = 0$, then the resulting polynomial is given by:

$$M_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \tag{17.6}$$

or put in the compact form:

$$M_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)x^j}{j!} \tag{17.7}$$

is called an n^{th} order Maclaurin polynomial.

The general form of the Lagrange Interpolating Polynomial of degree n is:

$$P_n(x) = \sum_{j=0}^{n} L_{n,j}(x)y_j \tag{17.8}$$

where

$$L_{n,j}(x) = \prod_{i=0, i\neq j}^{n} \frac{(x-x_j)}{(x_j-x_i)}. \tag{17.9}$$

Taylor polynomials approximate the value of a function in the neighborhood of x_0. Lagrange interpolating polynomials approximate a function very well on the entire interval. Secondly, the function which is being approximated need not be continuously differentiable to be approximated using Lagrange interpolating polynomials, but continuous differentiability is required in the case of approximations using Taylor Polynomials.

In the next sections, we shall discuss Lagrange Interpolation Polynomials and Hermite Interpolation Polynomials and derive approximations of the given elliptic functions. The polynomial approximations are then evaluated on the given interval.

18. Lagrange Interpolating Polynomials

A Lagrange Interpolating Polynomial of degree 5 is given by:

(18.1) $$P_5(x) = L_{5,0}(x)y_0 + L_{5,1}(x)y_1 + L_{5,2}(x)y_2 + L_{5,3}(x)y_3 + L_{5,4}(x)y_4 + L_{5,5}(x)y_5$$

where

(18.2) $$S = (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$$

is a known set of ordered pairs. From eq. (18.1), we have the following:

(18.3) $$L_{5,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}$$

(18.4)

(18.5) $$L_{5,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}$$

(18.6)

(18.7) $$L_{5,2}(x) = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}$$

(18.8)

(18.9) $$L_{5,3}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}$$

(18.10)

(18.11) $$L_{5,4}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}$$

(18.12)

(18.13) $$L_{5,5}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}.$$

Let us state here that a Lagrange Interpolating Polynomial of any degree can be chosen. We have chosen a 5^{th} degree here for ease of computation.

19. Solution Steps

Given the integral:

(19.1) $$\Psi = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{1}{3} - k^2\sin^2(\theta) + \frac{1}{15}\theta}} d\theta,$$

we have that $a = 0$ and $b = \pi/2$ and therefore we have the values from the table below.

	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5
θ	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$
$f(\theta)$	1.7321	1.6892	1.6634	1.644	1.6202	1.585

For the Table above, we have the polynomial approximation:

(19.2) $$P_5(x) = L_{5,0}(x)y_0 + L_{5,1}(x)y_1 + L_{5,2}(x)y_2 + L_{5,3}(x)y_3 + L_{5,4}(x)y_4 + L_{5,5}(x)y_5.$$

The above polynomial of degree 5 gives us:

$$P_5(x) = 0.01011472x^5 - 0.0319299x^4 - 0.0223331x^3$$
$$0.1248986x^2 - 0.17252619x + 1.732050807. \quad (19.3)$$

Upon integration, we get:

$$\int_0^{\frac{\pi}{2}} P_5(x)dx = 2.599475584. \quad (19.4)$$

20. Hermite Interpolating Polynomials

The general form of the Hermite Interpolating Polynomial is given by:

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

where

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x), \quad \hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x). \quad (20.2)$$

Also, we have:

$$L_{n,j}(x) = \prod_{i=0, i\neq j}^{n} \frac{(x - x_i)}{(x_j - x_i)}. \quad (20.3)$$

We shall now construct a Hermite Polynomial of degree 5 by constructing Lagrange polynomials of degree 3. We have that:

$$L_{3,0}(x) = -(36)\frac{(x - \pi/6)(x - \pi/3)(x - \pi/2)}{\pi^3} = -\frac{36}{\pi^3}(x^3 - 2.61799x^2 + 3.01571x - 0.86129)$$

$$L_{3,1}(x) = (108)\frac{(x)(x - \pi/3)(x - \pi/2)}{\pi^3} = \frac{108}{\pi^3}(x^3 - 2.09440x^2 + 1.64493x)$$

$$L_{3,2}(x) = -(108)\frac{(x)(x - \pi/6)(x - \pi/2)}{\pi^3} = -\frac{108}{\pi^3}(x^3 - 1.04720x^2 + 0.82247x)$$

$$L_{3,3}(x) = (36)\frac{(x)(x - \pi/6)(x - \pi/3)}{\pi^3} = \frac{36}{\pi^3}(x^3 - 1.04720x^2 + 0.54831x) \quad (20.4)$$

where

$$L'_{3,0}(x) = -\frac{36}{\pi^3}(3x^2 - 5.23593x + 3.01571) \quad (20.5)$$

$$L'_{3,1}(x) = \frac{108}{\pi^3}(3x^2 - 4.18880x + 1.64493) \quad (20.6)$$

$$L'_{3,2}(x) = -\frac{108}{\pi^3}(3x^2 - 2.09440x + 0.82247) \quad (20.7)$$

$$L'_{3,3}(x) = \frac{36}{\pi^3}(3x^2 - 2.09440x + 0.54831) \quad (20.8)$$

and $H_{j,k}(x)$ is given by

$$H_{3,0}(x) = [1 - 2(x - x_0)\left(-\frac{36}{\pi^3}\right)(3x^2 - 5.23593x + 3.01571)] \left(-\frac{36}{\pi^3}(x^3 - 2.61799x^2 + 3.01571x - 0.86129)\right)^2$$

$$H_{3,1}(x) = [1 - 2(x - x_1)\left(\frac{108}{\pi^3}\right)(3x^2 - 4.18880x + 1.64493)] \left(\frac{108}{\pi^3}(x^3 - 2.09440x^2 + 1.64493x)\right)^2$$

$$H_{3,2}(x) = [1 - 2(x - x_2)\left(-\frac{108}{\pi^3}\right)(3x^2 - 2.09440x + 0.82247)] \left(-\frac{108}{\pi^3}(x^3 - 1.04720x^2 + 0.82247x)\right)^2$$

$$H_{3,3}(x) = [1 - 2(x - x_3)\left(\frac{36}{\pi^3}\right)(3x^2 - 2.09440x + 0.54831)] \left(\frac{36}{\pi^3}(x^3 - 1.04720x^2 + 0.54831x)\right)^2$$

and

$$\hat{H}_{3,0}(x) = (x - x_0)\left(-\frac{36}{\pi^3}(3x^2 - 5.23593x + 3.01571)\right)$$

$$\hat{H}_{3,1}(x) = (x - x_1)\left(\frac{108}{\pi^3}(3x^2 - 4.18880x + 1.64493)\right)$$

$$\hat{H}_{3,2}(x) = (x - x_2)\left(-\frac{108}{\pi^3}(3x^2 - 2.09440x + 0.82247)\right)$$

$$\hat{H}_{3,3}(x) = (x - x_3)\left(\frac{36}{\pi^3}(3x^2 - 2.09440x + 0.54831)\right). \tag{20.9}$$

Using the above, we obtain the Hermite polynomial approximation:

$$H_7(x) = 0.0015x^7 - 0.0043x^6 + 0.0175x^5 - 0.0307x^4 - 0.0271x^3 - 0.6759x^2 - 0.1732x + 1.7321. \tag{20.10}$$

On integration of the above polynomial, we get:

$$\int_0^{\frac{\pi}{2}} H_7(x)dx = 2.59. \tag{20.11}$$

The results here indicate the integrals obtained by integrating the Hermite polynomials obtained as approximations of the continuous functions are more accurate than that of the integrals obtained from Lagrange or Taylor polynomial approximations of the same functions.

21. Acknowledgements

We thank Dr. Zephyrinus C. Okonkwo for serving as our project advisor and mentor. We also thank Albany State University's Office of Undergraduate Research and Albany State University's Title III Program for funding this research.

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ABSTRACTS

Reginald D. Bailey, Department of Mathematics, Morehouse College
The Role of the Vandermonde Matrix in Hermite Interpolatory Polynomials
Advisor: Dr. Benedict Nmah
In the paper "The Generalized Vandermonde Matrix", Kalman stated the Vandermonde determinant of order n as an identity. In this paper I use the Principle of Mathematical Induction to establish the identity. I also discuss the role of the Vandermonde Matrix in Hermite interpolatory polynomials.

Anthony J. Baker, Department of Mathematics, Sciences and Technology, Paine College
A Greedy Approximation Algorithm for Minimum CDS in Multihop Wireless Networks with Disparate Communication Ranges
Advisor: Dr. Lixin Wang
Multihop wireless networks (e.g., wireless ad hoc networks and sensor networks) have been widely used in various civilian and military applications. Unlike wired networks or cellular networks, no physical backbone infrastructure is installed in wireless networks. A communication session is achieved either through a single-hop radio transmission if the communication parties are close enough, or through relaying across intermediate nodes otherwise. Although a wireless network has no physical backbone infrastructure, a virtual backbone can be formed by the nodes in a connected dominating set (CDS) of the corresponding graph. Such a virtual backbone plays a very important role in routing, broadcasting and connectivity management in multihop wireless networks. To simplify the connectivity management, it is desirable to find a minimum connected dominating set (MCDS) of a given set of nodes. Since finding an MCDS for a graph is NP-hard, only distributed approximation algorithms in polynomial time are practical for multihop wireless networks. The MCDS problem has been studied extensively in multihop wireless networks with the unit-disk model where all nodes have uniform communication ranges. However, in practice the networking nodes may have different communication ranges either because of the heterogeneity of the nodes, or due to interference mitigation, or due to a chosen range assignment for energy conservation. With such a generalized communication model, two nodes can communicate with each other if they are within each others communication range. In this paper, we present a greedy approximation algorithm for computing a MCDS in multihop wireless networks with disparate communications ranges and prove that its approximation ratio is better than the best one known in the literature. Our analysis utilizes an improved relation between the independence number and the connected domination number.

Marcus Bartlett, Department of Mathematics, Clayton State University
On the Wiener Index of a Graph
Advisor: Dr. Elliot Krop
The Wiener index of a graph G is defined to be the sum of distances between every pair of vertices of G. When G is a k-ary tree, Hua Wang found a surprising relation between this index and the sum of distances between every pair of leaf vertices of G (called the

gamma index) and showed a counterexample for another conjectured functional relationship. In this talk, we define two new natural indices (the spinal index and the Bartlett index) which when summed with the gamma index above, yield the Wiener index. We then show analogous relations to that of Wang, produce a counterexample to a functional relation for the spinal index, and state a conjecture about the Bartlett index.

Jordan Campbell, Department of Mathematics, Morehouse College
Vibrant Colorings
Advisor: Dr. Rodney Kerby
Game theory is the name given to a wide array of mathematical processes that analyze a set of deliberate circumstances, including rules, challenges, and player interaction. The goal of game theorists is to predict the events and outcomes of such games. This presentation will examine a single player game called Vibrant Colorings, in which one player is given a nodule map or grid of $m \times n$ size and is told to color the map or grid in such a way that any four nodes that constitute a sub-rectangle of the entire map or grid are not the same color when given a certain number of colors k. The presentation will explain how to determine a winning strategy as well as how to predict the eventualities of rectangles of size $m \times n$ colored with k colors.

Corey Copeland, Department of Mathematics, Morehouse College
Analytical Solution of the Wave Equation in One, Two, and Three Dimensions
Advisor: Dr. Tuwaner Lamar
The area is Linear Algebra, Vector Analysis, and Boundary Value Problems. Background of the Fourier series is named in honour of Joseph Fourier. Fourier made important contributions to the study of trigonometric series. Fourier introduced the Fourier Series for the purpose of solving the heat equation in a metal plate. Fourier published his initial results in his 1807 Mmoire sur la propagation de la chaleur dans les corps solides (Treatise on the propagation of heat in solid bodies), and publishing his Thorie analytique de la chaleur in 1822. Research objectives are be able to define one dimensional space, two dimensional space, and three dimensional space in order to explain the Fourier Series. Research outcomes be able to define one dimensional space, two dimensional space, three dimensional space and Fourier Series in great details. I have proved the heat equation. Now I am working on how to prove the wave equation.

Joel Coppadge, **Sterline Caldwell** (**North Carolina A** & **T**),
and Sayomi Stallings (**U. of the District of Columbia**), Department of
Mathematics, Morehouse College
Two Color Nim
Advisor: Dr. Kenneth Berg
Nim is a mathematical game of strategy where red tokens are placed on a table in one or more stacks and two players take turns removing tokens. Each turn a player may remove tokens from only one stack and must remove at least one token. The player who removes the last red token wins the game. The winning positions known as Class W can be determined using an algorithm based on binary representations of heights of the stacks. No matter how many stacks there, the winning positions and winning moves can be found. Otherwise, it is a losing position known as Class L. The research our group did based on this game is called Two Color Nim which introduces a blue token(s) in the game. The rules for the game

and how to remove tokens remain the same. The winner of this game is who ever takes the last red token regardless of how many blue tokens are left. The goal of the research is to find all the winning positions and moves of two color Nim. We studied two different cases in our research. The first case was two stacks of red tokens with arbitrary heights with also one stack of blue tokens with an arbitrary height. The second case was having arbitrarily many stacks of red tokens with a maximum height of 2 and one stack of blue tokens of an arbitrary height. We completely solved both cases by determining all the winning positions and winning moves. We were able to find the winning moves by writing out different scenarios by hand or using a formula involving a matrix. This is ongoing research so other cases remained unsolved.

Thomas Coverson, Department of Mathematics, Morehouse College
Analysis of Lorenz System and Data Assimilation Techniques
Advisor: Dr. Rudy L. Horne
Edward Norton Lorenz (May 23, 1917 - April 16, 2008) was an American mathematician and meteorologist, and a pioneer of chaos theory. He discovered the strange attractor notion and coined the term butterfly effect. He also studied two dimensional convection in a horizontal layer of fluid heated from below. These studies resulted in the Lorenz System. The Lorenz System is made up of three partial differential equations:

$$\frac{dx}{dt} = \sigma(y-x), \ \frac{dy}{dt} = rx - y - xz, \ \frac{dz}{dt} = xy - bz$$

where r, b and σ are given parameters. In this research, these equations will be examined to find the fixed (critical) points of the Lorenz System, determine the stability and instability of the fixed points, calculate the numerical solutions of the Lorenz System using a computer program called Matlab, and applying data assimilation to the Lorenz System.

Joseph Crawford, Department of Mathematics, Morehouse College
Modular Representations of Graphs
Advisor: Dr. Reza Akhtar
A graph G has a representation modulo r if there exists an injective map $f : V(G) \rightarrow 0, 1, , r-1$ such that vertices u and v are adjacent if and only if $f(u)$ - $f(v)$ is relatively prime to r. The representation number of G, rep(G), is the smallest integer such that G has a representation modulo r. In this study, we look at the representation numbers of various graphs, such as the complete ternary tree and Harary graphs.

Nii Dodoo, Department of Mathematics, Morehouse College
Eventual Positivity for Special Types of Adjacency Matrices
Advisor: Dr. Ulrica Wilson
A matrix is strongly eventually nonnegative if it is eventually nonnegative and it has a power that is both irreducible and nonnegative. In 2010, Hogben introduced and used eventually r-cyclic matrices to establish an algorithm to deter- mine whether a matrix is strongly eventually nonnegative. The project investigates the eventual 2-cyclicity of special classes of cyclic characteristic matrices.

Dwayne Dorsey Jr., Department of Mathematics, Morehouse College
Enumeration of Objects Using Polya's Theorem

Advisor: Dr. Duane Cooper

In this project, we discuss the enumeration of molecules and other objects using combinatorial mathematics/group theory. More specifically, we use Polya's Fundamental Theorem, which analyzes permutations and equivalence relations to determine the number of non-equivalent arrangements of objects. Important to the development of Polyas Theorem is Burnside's Lemma which computes the number of equivalence classes into which a set is divided by an equivalence relation.

Lauren Dunn, **Shurron Green and Gerald Nicholson**, Department of Mathematics and Computer Science, Albany State University

Computer Security Vulnerability Prediction Algorithms: Is Algorithmic Information Theory a way forward?

Advisor: Dr. Robert Steven Owor

In todays sophisticated computer networks, information security is accomplished by the application of several methods to mitigate against information security breaches. These methods include among others: cryptography, authentication through user identification and passwords; antivirus programs, firewalls and challenge response systems; biometric systems, smart cards and intrusion detection systems as well as organizational rules policies and procedures. While these methods are commendable and have achieved success in defending against attacks, there is no known method to predict unknown future attacks. This paper proposes the use of Algorithmic Information Theory as a possible security solution in the design, implementation and deployment of information systems, so that inherent self-awareness is created within computer network systems in order to protect against attacks. The papers examines the foundational mathematical concepts in Algorithmic information theory as outlined by Kolgomorovs Complexity, Shannons Laws of Communications and Chaitans conjectures of decidability based on Gdels theorems and the Turing Computation thesis. These principles are applied in information creation, encryption and decryption during storage and transmission. Measures are proposed for vulnerability detection using Diophantine equations as an example.

Emily Fredericksen and Jessie Mayne, Department of Mathematics and Computer Science, Birmingham-Southern College

A Twisted Game: To Knot or Not to Knot on a new knot family

Advisor: Dr. Douglas Riley

Knot games can be used to illuminate properties of knots. Students at the SMALL REU at Williams College developed a game called To Knot or Not to Knot in which two players take turns determining the crossing information on a knot projection. In this paper, the game To Knot or Not to Knot is played on a knot family that consists of a clasped trefoil and k twists. A winning strategy has been found for the first player on members of this knot family with even twists. It is shown that the first player has a winning strategy due to symmetrical properties if they play their first move on the center position of the knot.

Tranisha Guthridge, Department of Mathematics, Clayton State University

A Survey of Graceful Labelings

Advisor: Dr. Christopher Raridan

We present a history of graph labeling, with a focus on graceful labelings of graphs, by providing a summary of many of the important papers. Also, we will present some of the more important theorems and conjectures concerning graph labelings.

Keywords: Graph labeling, graceful labeling, survey, history.

George Hagler, School of Mathematics, Georgia Institute of Technology
Incompleteness of Mathematics
Advisor: Dr. Enid Steinbart
Mathematics is in essence applied logic: we begin with axioms, such as the Axioms of Logic or the definitions of numbers, and proceed to show that other truths logically follow the assumption of these axioms. A mathematician named Hilbert proposed a question about the foundations of mathematics: can all the mathematical truths of the universe be reduced to some set of axioms? In other words, if we assume some countable number of axioms, could we then prove everything? For many years, the expected answer was yes. However, Kurt Godel proved the two Incompleteness Theorems. These theorems answered the problem negatively: for sufficient logical systems, it is impossible to ever reduce all mathematical (and, thus, scientific) truths to any countable set of axioms. In addition, no such system could prove itself to be consistent (to have no self-contradictions). The method that Godel proved this was by assigning the axioms and logical operators numbers and concatenating them with the separator zero, so that each proposition (and each proof) would be represented by a number. Using this Godel numbering system, he proceeded to show that, no matter what axioms you choose, there will always be some proposition that can be proven neither true nor false; thus, that proposition cannot be broken down into a logical structure. This is true for any such countable set of axioms, thus proving the result. In other words, the only theoretical mathematical system that could possibly prove all mathematical truths would have to have some uncountably infinite mass of axioms (or assume everything). You cannot define all of the real numbers until you define some uncountably infinite range of real numbers (such as the interval from 0 to 1); in the same way, you cannot logically show all mathematical truths until an uncountably infinite mass of truths is assumed.

James Handley, Department of Mathematics, Morehouse College
Analysis of the Short Pulse Equation
Advisor: Dr. Rudy L. Horne
The short pulse equation:

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

was recently derived by Schäfer and Wayne to model ultrashort optical pulses. In this presentation, we will derive a one-loop soliton solution for the short pulse equation (spe) and examine this solution given certain parameter values. Also, we will solve the spe using an exponential time-differencing numerical method.

Dieudonna Harris, **Maurice Gibson and Kheri Hicks**, Department of Mathematics and Computer Science, Albany State University
A Study of some Statistical Control Problems arising in Healthcare: Simmulations
Advisor: Dr. Zephyrinus C. Okonkwo
Statistical Process Control (SPC) has many and varied applications in manufacturing systems, service sectors, and any sectors where quality control and optimization are essential.

SPC applications have even become more important today as quality control is essential for cost minimization, profit maximization, customer satisfaction, competitive advantage, and enhancement of product market share. This paper focuses on statistical process problems arising in health: statistical process control problem for outpatient doctor visits, statistical process control problems for dental visits, and statistical process control problems for obstetrics and gynecological visits. Solutions to these problems have ramifications on efficient scheduling of patients, optimal use of resources and infrastructure, and optimal use of manpower resources at care centers, and other trade-off issues. Multifaceted optimal activities jointly impact on improvement of services, minimization of cost, improvement of the image of the hospitals, clinics, and care centers, and insurance companies whose goal is to attract more customers in order to increase their profit margins. We present simulations, run charts, and interpret results.

Hope Harris and Alicia Plotky, Department of Mathematics, Birmingham-Southern College
Disentanglement Puzzles in Knot Theory
Advisor: Dr. Douglas Riley
Given the four disentanglement puzzles Cowboy's Hobble, Coiled Again, Heart String and Three Triangles, we try to solve them topologically using sequences of Reidemeister moves. This proves to be quite difficult since the constraints on the puzzles are geometric which do not apply to knots and links in topology. In order to do this, we model the puzzles using knot diagrams and perform Reidemeister moves to produce the unlink. We show that any solution to Cowboy's Hobble requires the ring-hook move and we generalize Coiled Again to any number of coils which will require the same amount of twists as the number of coils to solve the puzzle. In addition, we show that one end of the string on the Heart String must be free in order to obtain the unlink. We also show the solution to Three Triangles and how it is similar to the Heart String.

Terry Henderson and Aaron Ostrander, Department of Mathematics and Computer Science, Berry College
Exponential Domination of Triangular Grid Graphs
Advisor: Dr. Jill Cochran

A number of real-world networks can be modeled using graphs whose vertices exert influence over one another. These networks are usually treated in terms of *domination* of graphs, and recently work has been done where the domination that one vertex can have on another decreases exponentially with the distance between the two vertices. We consider the case where a *dominating vertex*, v, contributes a factor of $2^{1-d(u,v)}$ to the *domination* of u for $u \neq v$. We say $S \subset V(G)$ *exponentially dominates* G if every vertex in $V(G) \setminus S$ has domination greater than 1. The *exponential dominating number* of G, $\gamma_e(G)$, is the least number of dominating vertices needed to exponentially dominate a graph. We say $S \subset V(G)$ *totally exponentially dominates* G if every vertex in $V(G)$ has domination greater than 1. The *total exponential dominating number* of G, $\gamma_{te}(G)$, is the least number of dominating vertices needed to totally exponentially dominate a graph. We explore $\gamma_e(G)$

and $\gamma_{te}(G)$ for triangular grid graphs, providing exact values and methods for generating inequalities.

Anthony Hicks, Department of Mathematics and Computer Science,
Albany State University
Cryptography Techniques in Computer Networks
Advisor: Dr. Khalil Dajani
Secure communication link has widely become the most important method of todays modern society and their developments are increasing dramatically. The use of secure link has relied on the confidentiality and security of its data transmission. The emergence of e-commerce including electronic funds transfer, internet marketing, online transaction processing and electronic data interchange are widely used to serve the convenience to users. The communication between both user and system administrator using insecure (public) channel to exchange data are easy enough to the intruders who wish to get the information about the exchanged data. This paper explores current techniques and protocols of cryptography and their opportunities and challenges that are concerned with the security and authenticity of exchanged messages for secured communication. Detection schemes for conventional communication and insecure networks as well as protocols efficiency and implementation are discussed.

Kristopher Jones, Department of Mathematics, Morehouse College
Voting in Agreeable Societies
Advisor: Dr. Duane Cooper
Voting can be expressed mathematically in a one-dimensional or multi-dimensional space. Platforms and voters are represented by points in a space, more accurately, referred to as a spectrum or policy space. The focus of my research is on one-dimensional or linear societies. Foundational knowledge of set theory, analysis, and graph theory is important for full comprehension of the research concepts, theorems and proofs presented. My objective is to provide proofs of the Agreeable Linear Society Theorem and Super-Agreeable linear society theorem along with others to answer the following question: Given any political spectrum when can we guarantee that some fraction of the population will agree on some candidate or platform? Concepts from graph theory are applied to describe voting in linear societies and to prove additional theorems.

Sean Laster, Department of Mathematics, Morehouse College
Convergence of the Fourier Series
Advisor: Dr. Joseph Eyles
Joseph Fourier introduced the series that now carries his name in order to solve the equation for heat transfer in a metal plate. A Fourier Series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. This paper will examine and explain the behavior of the Fourier Series when it meets the Dirichlet Conditions. Under the Dirichlet Conditions, the Fourier series converges to f(x) if x is a point of continuity and converges to some value if x is a point of discontinuity.

Biliaminu Lawal, Department of Mathematics, Morehouse College
Biostatistics of Populations of Cancerous Cells

Advisor: Dr. Chuang Peng

This paper in Biostatistics proves how the chi-squared and normal distributions are derived for each degree of freedom using the beta function. The paper also uses the Odds and Risk Ratio to prove the Wald and Likelihood tests of Homogenity through the Hypothesis Test. It first uses a reasonable sample population to properly test the experiment; then extends, under these conditions, to a larger sample population to calculate the sample mean. The paper then tests the hypothesis under a 95% interval and rejects or accepts the hypothesis depending where the data lies. The data is graphed using the chi-square distribution, which is the normal distribution squared, and was obtained by using the Z-test. These distributions are representations of the data and are determined by the degrees of freedom. The degrees of freedom are derived from the gamma and beta functions, where alpha and beta are greater than zero and are uniform distributions.

Herns Mesamours, **Kelvin Williams and Clarence Spearman**,
Department of Mathematics and Computer Science, Albany State University

Approximating Solutions of Nonlinear Differential Equations using the Runge-Kutta-Fehlberg Method

Advisor: Dr. Zephyrinus C. Okonkwo

In this paper, we study the Runge-Kutta-Fehlberg method, a numerical technique used to find approximate solutions to nonlinear first order ordinary differential equations. Examples are presented for illustration. The classical Runge-Kutta Method of Order 4 (RK4) is applied to the same initial value problems and the results of both methods are compared. Furthermore, error analyses of these methods are discussed.

Michael Ngo, Department of Mathematics, Clayton State University

On new bounds for the monophonic number of Cartesian products of graphs

Advisor: Dr. Elliot Krop

Given two vertices u; v in a graph G, a chordless path from u to v is also known as a monophonic path. Let JG[u; v] be the monophonic closed interval consisting of all vertices on all monophonic paths from u to v. For any subset of S of vertices of G, let JG[S] be the set of all monophonic intervals for every pair of vertices from S. A set S is called a monophonic set of G if JG[S] is the set of all vertices in G. The minimum cardinality of S, for all S subsets of vertices of G, so that JG[S] is a monophonic set, is called the monophonic number of G and is denoted mn(G). In this talk, we further describe these concepts, beginning with an introduction of basic graph theoretic terminology. We then discuss some new bounds discovered by A.P. Santhakumaran and S.V. Ullas Chandran on the monophonic number of graphs C which are Cartesian products of two graphs G and H. Finally, we mention some open problems in the subject. This talk will be self-contained and aimed at a general audience.

Keywords: monophonic path, monophonic set, monophonic number, Cartesian product of graphs

Huda Qureshi, Department of Mathematics, Birmingham-Southern College

The Easy Configurations of the Lights Out Cube Game

Advisor: Dr. Douglas Riley

Lights Out was an electronic game released by Tiger Toys in 1995. The object of the game was to turn off a given light configuration, under a certain set of restrictions. An “easy” configuration of the game is one in which turning off all the initially lit lights on the Lights Out game turns off all the lights and wins the game. Through Linear Algebra techniques, the research of Bruce Torrence and Jacob Tawney, and MATLAB code, I was able to find the easy configurations of the Lights Out Cube to be 2^{4n} possible easy configurations for a cube game of side length n.

Virginia Seale and Hannah Wirey, Department of Mathematics, Birmingham-Southern College
Whispering Jokers and Their Secrets
Advisor: Dr. Douglas Riley
Mathematics is often used to examine the functionality of card tricks. We aim to mathematically examine a family of tricks involving reference cards and generalize a specific trick to an infinite number of cards and an infinite number of reference cards. The trick examined in this paper is called Whispering Jokers. In the process of examining this specific card trick we were able to develop mathematical formulas to represent the two permutations performed in this trick. It was found that Whispering Jokers could be generalized to an infinite number of cards while still using two reference cards. In generalizing this trick to an infinite number of reference cards, the permutations were studied and it was found that the trick could only be generalized to an even number of reference cards but not odd. In proving this, we were able to show that even though a spectator may handle the cards during the trick, the mathematics behind the trick is actually in control.

Melanie Short and Nino Christopher Yu Tiamco, Department of Mathematics, Birmingham-Southern College
Primitive Right Triangles and Their Geometric Properties
Advisor: Dr. Douglas Riley
We investigate geometric properties of primitive right triangles. We begin by outlining results concerning a triangle's semiperimeter and relating a triangle's area to its perimeter. We conclude by proving that for every natural number n where n has k distinct odd primes in its canonical factorization, there exist exactly 2^k distinct primitive right triangles whose area equals n times its perimeter.

Mike Stonewall, Department of Mathematics, Birmingham-Southern College
Using Natural Cubic Splines to Model Local Landform
Advisor: Dr. Douglas Riley
Little is known about calculations used to describe the Birmingham-Southern College area of land mass. We confirm the accepted figure of 192 acres by utilizing simple land surveying techniques. First, we collect geographical location data via smart phone remote sensing. Next, we use natural cubic splines to interpolate the campus boundary. And finally, we integrate across a bounded region. Missing our mark by approximately 3 percent shows the accuracy of natural cubic splining as an effective method for line interpolation.

Ashley Sumter, Department of Mathematics and Computer Science,
Albany State University
Samples with a Fixed Probability of an Event
Advisor: Dr. Li Feng

In this paper, we tackle the following problem. Suppose a population contains two different types of objects, say, black marbles and white marbles. Given that the probability of drawing two marbles without replacement and getting two black marbles is p, find all of the samples in which the probability of such an event (that is, drawing two marbles without replacement and getting two black marbles) remains the same. We will show that the problem is equivalent to find all the triangular numbers that their multiples are also triangular numbers. We present the complete solutions for p=1/2 and p=1/3 as examples.

Maxalan Vickers, Department of Mathematics, Morehouse College
Linear Geometric Constructions: The How, What, and Why, of Constructing Them
Advisor: Dr. Lawrence Smolinsky
Our research will discuss and give examples of two ways to create geometric constructions and the tools needed to produce these constructions. One way uses a straightedge and compass, while the other uses a straightedge with two notches and a compass, otherwise known as the neusis construction. Also, we will discuss those numbers that are constructible using these two ways, such as the square root of 2, and those that we cannot currently construct, such as the fifth root of a number, a where a is a constructible number. Last but not least, the research also introduces the key terms, fields and towers, and establishes connections between these key terms and our linear geometric constructions.

Charles Watts, Department of Mathematics, Morehouse College
A Stunning Theorem
Advisor: Dr. Ulrica Wilson
A square matrix $A = (a_{ij})$ is positive, denoted $A > 0$ if $a_{ij} > 0$ and eventually positive if there exists k_0 such that for all $k \geq k_0$, $A^k > 0$. Eventually positive matrices were introduced in 1978 by S. Freidland. We present some results about the eventual positivity of adjacency matrices of several classes of graphs including cycles and zero divisor graphs.

Charles Wilkes, Department of Mathematics, Morehouse College
2-1 Achievement Game
Advisor: Dr. Curtis Clark

Let F be a graph with no edges. The 2-1 achievement game, namely F, is a two-player game that moves alternately. The board consists of n vertices. Player A moves first connecting two vertices, making an edge. Player B connects two different vertices, making an edge. Players then alternate moves. The graph F is achievable on Kn if Player A can successfully achieve graph F. The minimum n such that F is achievable on Kn is its 2-1 achievement number, a2 (F). The minimum number of move to achieve F, donated m2 (F) is the least number of moves played by Player A to make F on the complete graph with a2 (F) vertices. It was proved that a2 (C5)=5 and that m2 (C5)=5.

Kelvin Williams and Clarence Spearman, Department of Mathematics and Computer Science, Albany State University
Hermite and Cubic Spline Polynomial Interpolation of Perturbed Elliptic Functions and Their Integrals

Advisor: Dr. Zephyrinus C. Okonkwo
In this paper we examine derive Hermite and Cubic Splines polynomial approximations of perturbed elliptic functions and subsequently evaluate their integrals on closed intervals of the real line. Continuity of such elliptic functions is assumed but obtaining closed form integrals solutions of such integrals are challenging, hence, adoption of numerical integration techniques will be in order. The results are compared with Taylor polynomial approximation and Maclaurin polynomial approximation obtained from a Binomial series generated from the elliptic integrals. Using Weierstrass approximation theorem, the absolute error is computed in each case. Examples are presented for illustration.

Priyanka Yarlagadda, Department of Mathematics, Sciences and Technology,
Paine College
Logistic Equation Using Microbiological Parameters with an Application to Bioremediation
Advisors: Dr. C.P. Abubucker, Dr. C. R. Nair and Dr. Bibekananda Mohanty
Microorganism has remarkable ability to degrade environmental pollutants. Certain microbes, like Bacillus Sphaericus use Cr- VI as a source of energy and efficiently convert this hazardous chemical into less harmful Cr-III. Microbial capabilities for biodegradation vary widely. Many mathematical models are used to describe the microbial growth. One such model is the logistic equation. This equation contains the mathematical parameters a, b and c. In this paper we explore how these abstract mathematical parameters are related to the microbiological parameters, maximum specific growth , lag time and the asymptotic value A. These relations are used to derive an equation involving , and A from the logistic equation. The derived logistic equation using the microbiological parameters is . This model is applied to the data collected from the experiment by the authors through a grant funded by the United States Department of Energy. A study was undertaken through this grant to examine the potential of Bacillus sphaericus in the reduction of hexavalent chromium in a liquid suspension culture. Bacillus sphaericus cells were grown in a liquid fermentation medium containing sodium acetate, yeast extract, magnesium chloride, and calcium chloride. The kinetic parameters of this bacterial growth were calculated and the growth was modeled by the logistic equation. [This study was supported, in part, by a grant from the Department of Energy awarded to Paine College.]