Proceedings of the Harriett J. Walton Symposium on Undergraduate Mathematics Research

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Preface

The Department of Mathematics initiated the Harriett J. Walton Symposium on Undergraduate Mathematics Research to encourage undergraduate students in mathematics research and practice. We believe that undergraduate research experiences must count among the most challenging and rewarding experiences for college students. To research and present material beyond what is traditionally covered in the classroom to explore mathematics independently may be considered the best career preparation for students regardless of their post-college plans. This yearly symposium on undergraduate research is dedicated to Professor Harriett J. Walton who served forty-two years on the faculty of Morehouse College.

The Fifteenth Annual Harriett J. Walton Symposium on Undergraduate Mathematics Research, a regional undergraduate mathematics conference, was held on the campus of Morehouse College in Atlanta, Georgia, USA, on April 1, 2017. Forty-three students from Agnes Scott College, Albany State University, Birmingham-Southern College, Georgia Institute of Technology, Morehouse College and University of Georgia gave twenty-six presentations on their research and studies in mathematics and related fields. The Symposium was sponsored by the Department of Mathematics and the Division of Science and Mathematics of Morehouse College. This volume contains three articles and twenty-six abstracts submitted by the Symposium participants and their advisors.

The organizers of the Symposium thank the presenters and their advisors for preparing a remarkable collection of lectures for the Symposium. We thank the referees for their service to evaluate and improve the papers before their publication. We thank the administration of Morehouse College for their generous support, especially Dr. Duane Jackson, Chair of the Division of Science and Mathematics, Dr. Garikai Campbell, Provost and Senior Vice President for Academic Affairs, and Interim President Dr. William J. Taggart. Special thanks to Ms. Sandra Strickland for coordinating many aspects of the Symposium. Finally we thank Professor Walton for attending the Symposium.

Professor Harriett J. Walton

In September 1958, Harriett J. Walton joined the faculty of Morehouse College during the presidency of Benjamin Elijah Mays. She became a member of a team of three persons in the Department of Mathematics where she worked with the legendary Claude B. Dansby who served as Department Chair. Dr. Walton and her two colleagues taught all of the mathematics for the majors as well as the mathematics for non-science students. Dr. Walton relates that two of her favorite courses that she taught during this period were Abstract Algebra and Number Theory. The three-member mathematics department did an excellent job of preparing their mathematics majors for graduate school and the other students for success in their respective disciplines. In fact it was during this period of history that Morehouse gained the reputation of being an outstanding Institution especially for African American men. As the department grew, Dr. Walton shifted her attention away from mathematics majors and began to concentrate on students who needed special attention and care in order to succeed in mathematics. She became an advisor, mentor, tutor and nurturer to a large number of students matriculating at Morehouse College. Because of the caring attitude that she had for her students, some of them to this day refer to her as “Mother Walton.”

Dr. Walton has never been satisfied with mediocrity. Throughout her teaching career she demonstrated a love for learning. In 1958 when she arrived at Morehouse College she had an undergraduate degree in mathematics from Clark College in Atlanta, Georgia, a Master of Science degree in mathematics from Howard University, Washington D.C., and a second Master’s degree in mathematics from Syracuse University. While at Morehouse teaching full time and raising a family of four children, Dr. Walton earned the Ph.D. degree in Mathematics Education from Georgia State University. After receiving her doctorate, Dr. Walton realized the emerging importance of the computer in education so she returned to school and in 1989 earned a Master’s degree in Computer Science from Atlanta University. She is indeed a remarkable person.

Dr. Walton’s list of professional activities, awards and accomplishments during her career is very impressive and too lengthy to be enumerated here. However a few special ones are her memberships in Alpha Kappa Mu, Beta Kappa Chi, Pi Mu Epsilon, and the prestigious Phi Beta Kappa Honor Society. Additionally she was selected as a Fulbright Fellow to visit Ghana and Cameroon in West Africa. Dr. Walton’s professional memberships included the American Mathematical Society, the Mathematical Association of America, National Council of Teachers of Mathematics (NCTM) and the National Association of Mathematicians (NAM). She served as Secretary/Treasurer of NAM for ten years. In May 2000, Dr. Walton retired from Morehouse College after forty-two years of service.

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Graph Derangements and Permutations for Known Graphs

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Abstract: Let G be a simple graph. We consider a generalization of permutations and derangements on an n element set by defining graph permutations and graph derangements on the vertex set of G that takes into account the adjacencies and edge structure of G. We then count the number of graph derangements and permutations for the complete graph, K_n; the ladder graph, L_n; and the complete, balanced bipartite graph, $K_{n,n}$ while also considering the ratio of graph derangements and graph permutations and the asymptotic behavior. We show that the number of derangements on L_n is F_{n+1}^2, where F_n is the n^{th} Fibonacci number. We conclude the paper by conjecturing the ratio of interest is always less than or equal to 1/2.

1. Notation and definitions

Permutations and derangements of sets are known combinatorial objects, and they have a natural generalization to a general graph G. Recall that on a set $S = \{1, 2, ..., n\}$, a permutation is a way of arranging the n elements into an ordered list, and a derangement is a permutation where no element remains in its original position, i.e. 1 is not the first element in the list, 2 is not the second, etc. It is known that there are $n!$ permutations on S and approximately $n!/e$ derangements. Furthermore, the probability that a random permutation is a derangement is asymptotically 1/e. With these definitions in mind, we proceed to define graph derangements and permutations.

Definition 1.1. *[2, Section 2.1] Given a graph G, we say a permutation $\sigma : V(G) \to V(G)$ is a* graph permutation *if for $v \in V(G)$, either $\sigma(v) = v$ or $\sigma(v)$ is adjacent to v. We denote the number of graph permutations as $per(G)$.*

Definition 1.2. *Given a graph permutation $\sigma : V(G) \to V(G)$, we say $v \in V(G)$ is a* fixed point *if $\sigma(v) = v$.*

Now that we have a concept of graph permutations and fixed points, we are able to define graph derangements.

Definition 1.3. *[2, Section 2.1] A* graph derangement *is a graph permutation with no fixed points. Denote the number of graph derangements as $der(G)$.*

At this point, we illustrate these definitions with a couple of examples. Consider $\sigma_1 = (1234), \sigma_2 = (13)(24)$, and $\sigma_3 = (24)$ on C_4 and K_4 pictured below. We see σ_1 is a graph derangement on C_4 and K_4, σ_2 is a graph derangement on K_4, but neither a graph derangement nor graph permutation on C_4. Finally σ_3 is a graph permutation on K_4, but neither a graph permutation nor a graph derangement on C_4.

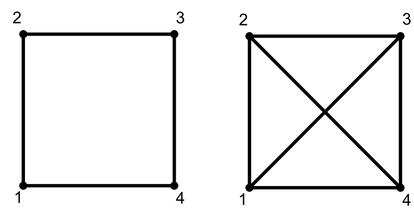

Figure 1 : C_4 (left) and K_4 (right).

As a final note about the definitions, if $G = K_n$, then $per(K_n) = n!$ since K_n is the graph corresponding to an n element set. Similarly, $der(K_n)$ is approximately $n!/e$, and thus as n goes to infinity we have that the ratio of graph derangements to graph permutations limits to $1/e$. In some sense K_n represents the least structured graph, as the image of a vertex is not restricted under graph permutations (and derangements). Thus a natural question is what is the ratio of graph derangements to graph permutations in a more structured graph? Is $1/e$ an upper bound or can we do better? We explore this question in the remainder of the paper by calculating the ratio for some families of graphs.

2. An Interesting Combinatorial Example: L_n

We'll now examine a graph whose derangements form an interesting combinatorial sequence. Consider $G = L_n$, the ladder graph with n rungs. For convenience, we'll enumerate the vertices of the ladder graph as follows.

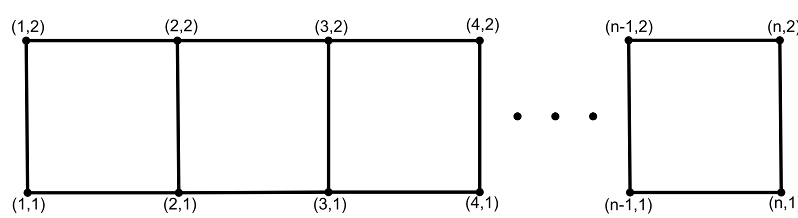

Figure 2 : The ladder graph, L_n, with vertices enumerated.

As a note, the k^{th} rung is the edge between $(k, 1)$ and $(k, 2)$. We'll now find an explicit formula for $der(L_n)$. To do this, we'll derive a recursion using the idea of faults; however, before we proceed, we make a definition.

Definition 2.1. *[1, p.7] Given L_n, we'll say that a graph permutation has a* fault (at rung k) *if neither of the vertices at rung k are mapped to rung $k+1$ and vice versa, and we'll say a graph permutation is* fault-free *if it does not contain any faults.*

As an example of this definition, consider L_2. We'd like to count the derangements of L_2. So we'll count the ones that are fault free, and then the ones that have a fault at rung 1. For L_2 to have a fault-free derangement, we need to interchange $(1, 1)$ and $(2, 1)$, ie $(1, 1) \leftrightarrow (2, 1)$, in which case we have a perfect matching; otherwise, we have the four-cycle $(2, 1) \to (2, 2) \to (1, 2) \to (1, 1)$. We also obtain a fault-free derangement if we reverse the orientation of the cycle, namely $(1, 1) \to (1, 2) \to (2, 2) \to (2, 1) \to (1, 1)$. For a graph derangement on L_2 to have a fault at rung 1, it must be the case that $(1, 1)$ and $(1, 2)$ interchange and likewise $(2, 1)$ and $(2, 2)$ interchange. Thus in total $der(L_2) = 3 + 1 = 4$.

We now proceed to calculate $der(L_3)$ through which will we see the general formula for $der(L_n)$. There are two fault-free derangements on L_3, the clockwise and counterclockwise 6-cycle. For derangements with a fault at rung 1, they contain the matching $(1, 1) \leftrightarrow (1, 2)$ together with a derangement of the remaining L_2 subgraph–there are four of these. Finally, there are 3 derangements whose first fault is at rung 2 since any derangement of this form must contain the matching $(3, 1) \leftrightarrow (3, 2)$ and a fault-free derangement of the remaining L_2 subgraph. Note if the derangements of the L_2 graph are not fault-free, then that means we have a fault at an earlier rung, and hence it has already been included in our count. In summary, $der(L_3) = 9$.

To find a general formula for $der(L_n)$, we proceed as above. There are two fault-free derangements, namely the two oriented $2n$-cycles. For derangements with a fault at rung 1, there are $der(L_{n-1})$, and for a fault first occurring at rung 2, there are $der(L_{n-2})$. Note that a derangement whose first fault is at rung k for $k \geq 3$ consist of a cycle on the L_k subgraph preceding the fault and a derangement on the remaining L_{n-k} subgraph. Thus letting $der(L_0) = 1$, we see

$$der(L_n) = der(L_{n-1}) + 3\,der(L_{n-2}) + 2\sum_{k=3}^{n} der(L_{n-k})$$

which is fairly cumbersome; however, since most of the terms have a coefficient of 2, we'll consider the difference $der(L_n) - der(L_{n-1})$ and by simplifying, we find

$$der(L_n) = 2\,der(L_{n-1}) + 2\,der(L_{n-2}) - der(L_{n-3}).$$

This sequence is 1, 1, 4, 9, 25, 64, 169, 441, ... which are the squares of the Fibonacci numbers.

We now count the graph permutations on L_n. So proceeding as above, we note that $per(L_1) = 2$ since we have a perfect matching and the identity permutation. For $per(L_2)$, we have four permutations with a fault at rung 1 and five fault-free permutations, and thus $per(L_2) = 2per(L_1) + 5per(L_0) = 9$. Now for $per(L_3)$, we have $2per(L_2)$ permutations with a fault at rung 1, $5per(L_1)$ permutations with a fault at rung 2 (but not before), and $4per(L_0)$ that are fault free, namely the two 6-cycles and two matchings that have a fixed point in opposite corners (if $(1,2)$ is fixed, then so is $(3,1)$). As before generalizing the $n = 3$ case, we define $per(L_0) = 1$ and find the general formula to be

$$per(L_n) = 2per(L_{n-1}) + 5per(L_{n-2}) + 4\sum_{k=3}^{n} per(L_{n-k}).$$

Using the same trick from above, we find

$$per(L_n) = 3per(L_{n-1}) + 3per(L_{n-2}) - per(L_{n-3}).$$

This recursion gives us the sequence $1, 2, 9, 32, 121, 450, 1681, \ldots$ (which is alternately a perfect square and twice a perfect square).

Finally, we calculate the asymptotic ratio of $der(L_n)$ and $per(L_n)$. So first note that the characteristic polynomial for $\{der(L_n)\}_{n=1}^{\infty}$ is

$$x^3 - 2x^2 - 2x + 1 = (x+1)\left(x - \left(\frac{3-\sqrt{5}}{2}\right)\right)\left(x - \left(\frac{3+\sqrt{5}}{2}\right)\right)$$

and for $\{per(L_n)\}_{n=1}^{\infty}$ is

$$x^3 - 3x^2 - 3x + 1 = (x+1)(x - (2-\sqrt{3}))(x - (2+\sqrt{3})).$$

So asymptotically $der(L_n)$ behaves like $\left(\frac{3+\sqrt{5}}{2}\right)^n$ and $per(L_n)$ is asymptotic to $(2+\sqrt{3})^n$ from which we see $\lim_{n\to\infty} \frac{der(L_n)}{per(L_n)} = 0$.

The authors would like to thank Curtis Herink for his observations and suggestions in simplifying this section.

3. An Upper Bound: $K_{n,n}$?

We have considered derangements and permutations for a couple of different types of graphs. Now we focus on the complete bipartite graph $K_{n,n}$ which we'll think of as $K_{n,n} = (V, E)$ where $V = V_1 \sqcup V_2, |V_1| = n$ and $|V_2| = n$. Note that V_1 and V_2 represent the two disjoint sets of vertices. As before, we first determine $der(K_{n,n})$, and then we proceed to determine $per(K_{n,n})$.

In order to find $der(K_{n,n})$, we'll consider bijections of an n element set to itself. It is well known that there are $n!$ such bijections. So if we think of V_1 and V_2 as two copies of the same n element set, and we note that due the structure of $K_{n,n}$ (and the definition of a graph derangement) the vertices of V_1 must map injectively to the vertices of V_2 and vice versa. So there are $n!$ ways to map V_1 to V_2 and similarly there are $n!$ ways to map the vertices of V_2 to V_1. Since the choice of mappings are independent, there are total of $(n!)^2$ ways to derange the vertices of $K_{n,n}$ and hence $der(K_{n,n}) = (n!)^2$.

Before we proceed, we must first prove a technical lemma.

Lemma 3.1. *Let $K_{n,n} = (V_1, V_2, E)$ and suppose $\sigma : V(K_{n,n}) \to V(K_{n,n})$ is a graph permutation. If there are j fixed points (of σ) in V_1, then there are j fixed points in V_2.*

Proof. Suppose we are given σ, and there is exactly one fixed point in V_1. Then there are $n-1$ vertices in V_1 to permute and n vertices in V_2 that permute. Then either the n vertices of V_2 map to vertices in V_1 or there are fixed points. If all n vertices map to V_1, then by the Pigeonhole Principle, at least two vertices map to the same vertex in V_1, but this can not happen since σ is a graph permutation. Thus it must be the case that at least one vertex in V_2 is fixed. Suppose that V_2 has l fixed points where $2 \leq l \leq n$. Then by a similar Pigeonhole argument, there are $l-1$ additional vertices in V_1 that are fixed. □

Now armed with this lemma, we can find $per(K_{n,n})$ by looking at the number of fixed points in V_1. If there are zero fixed points, then we have $der(K_{n,n})$ permutations. If there is exactly one fixed point, then by the lemma, there is also a fixed point in V_2. We can choose each of these fixed points in $\binom{n}{1}$ ways and the choices are independent which gives us a told of $\binom{n}{1}^2$ choices. By choosing our fixed points, we've restricted ourselves to deranging the $K_{n-1,n-1}$ for which we know there are $der(K_{n-1,n-1})$ possible ways to accomplish this. This gives a total of $\binom{n}{1}^2 der(K_{n-1,n-1})$ permutations with exactly one fixed point in V_1. If we generalize and consider permutations with $2 \leq j \leq n$ fixed points in V_1, then it is not hard to see that there are $\binom{n}{j}^2 der(K_{n-j,n-j})$ permutations with this property. Now summing over all possible numbers of fixed points in V_1, we find

$$per(K_{n,n}) = \sum_{j=0}^{n} \binom{n}{j}^2 der(K_{n-j,n-j}) = \sum_{j=0}^{n} \left(\binom{n}{j} (n-j)! \right)^2$$

.

Finally, we consider the asymptotic ratio of $der(K_{n,n})$ and $per(K_{n,n})$. Since the denominator involves a sum, we will instead consider $\lim_{n\to\infty} \frac{per(K_{n,n})}{der(K_{n,n})} = \sum_{j=0}^{\infty} \frac{1}{(j!)^2}$ which is approximately 2.2795; however, we're interested in the reciprocal of this quantity which is approximately 0.4387.

4. Conclusions

After exploring various examples, we are in a position to give a partial answer to what is the asymptotic behavior of derangements to permutations for a family of graphs. A sharp lower bound of 0 is achieved by L_n as demonstrated in section 3. The question then becomes what is an (sharp) upper bound. At first, $1/e$ appears to be a good candidate for an upper bound, but this is clearly not the case as we saw in section 4, the ratio of derangements to permutations of the complete bipartite graph limits to approximately 0.4387. So far, this is the highest value achieved by any of the family of graphs we have examined. We currently conjecture that $der(G)/per(G) \leq 1/2$ for simple, undirected graphs G with K_2 being the only graph to achieve the upper bound.

5. References

[1] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, USA, 2003.

[2] P. L. Clark, *Graph Derangements*, Open J. Disc. Math., 3 (2013) 183-191.

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Cyclic Dynamical Systems

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Abstract: In this paper, we study finite cyclic dynamical systems generated by the function f_r given by Adamaszek, Adams, and Motta in [1]. The authors in [1] identified a special class of points denoted q-swift points. The existence of at least one q-swift point was found to imply a single periodic orbit. We extend this definition to q_d-swift points which imply d periodic orbits. The authors proved in [1] that equidistant, "regular" systems are at the base of all cyclic dynamical systems. We further show that, given a unique orbit on an equidistant arrangement, adding a finite number k points into $(x_j, x_{(j+1) \bmod \ell})_{S^1}$ where $x_j, x_{(j+1) \bmod \ell}$ are two consecutive points of the arrangement, produces $(k+1)^2$ possible systems, each of which must be isomorphic to one of two regular systems.

1. Introduction

We study cyclic dynamical systems produced by a function f_r presented in [1] on a finite subset of points along a circle S^1 of unit circumference. Simply put, f_r maps points, x_i, from a subset $X \subseteq S^1$ to the clockwise-furthest element of X within some arc length $0 < r \leq 1$ on S^1 from x_i. Since our domain is a finite set of points, the function generates a finite dynamical system.

The initial motivation for studying f_r in [1] was that it can be described in terms of a Vietoris-Rips simplicial complex. The authors of [1] used the notation $\mathbf{VR}(X;r)$ to describe a Vietoris-Rips simplicial complex on a vertex set X where $\sigma \subseteq X$ is a simplex if the diameter of σ is less than r. The number of simplices on $\mathbf{VR}(X;r)$ can be exponential in $|X|$, and so the study of f_r is intended to simplify the behavior of $\mathbf{VR}(X;r)$ for the sake of improved understanding.

We would like to thank Dr. Rachel Bayless for her mentorship. In addition, we thank the Goizueta Foundation and the Frances Marx Shillinglaw Women in Science Endowment Fund for their contributions.

We study f_r in the context of the field of dynamics as a whole in order to relate the structure of a system to its dynamics. We identify features of the system's structure which allow us to understand the system's dynamics without detailing each iteration of the function. In [2], the authors created a classification for fixed point systems in order to better analyze the dynamics of models for applications of finite dynamical systems, namely those of biochemical networks, without having to specify each state transition. In the same light, we identify elements which signal a specific number of periodic orbits. We also study equidistant arrangements of points and describe properties of a single orbit on such an arrangement. In doing so, we gain further understanding of what we can say about the dynamics of a system given its structure.

After describing preliminary details of f_r in Section 2, we focus in Section 3 on an expansion of the q-swift points introduced in [1]. The authors showed in [1] that the existence of a q-swift point implies one periodic orbit with specific relationships between r and the length and winding number of the orbit. We define a q_d-swift point and show that the existence of one implies d periodic orbits and an analogous relationship between r, length, and winding number. In Section 4, we study equidistant arrangements. It was stated in [1] that the core of any cyclic dynamical system is isomorphic to some system with an equidistant arrangement. Here, we detail properties of single orbits on equidistant arrangements and enumerate the possible systems formed when we perturb such an arrangement slightly by inserting a finite number of points between two consecutive elements of the existing arrangement.

2. Preliminaries

Throughout this paper, we use the function f_r described in [1].

Definition 2.1. *Let $S^1 = \mathbb{R}/\mathbb{Z}$ and consider a finite subset $X \subseteq S^1$ with $0 < r \leq 1$. Note that S^1 is thereby the circle of unit circumference. Let $f_r : X \to X$ be defined such that $f_r(x) = \arg\max_{y \in X \cap [x, x+r)_{S^1}} \{\vec{d}(x, y)\}$. In other words, $f_r(x)$ maps x to the clockwise-furthest element of $X \cap [x, x+r)_{S^1}$.*

When discussing the physical arrangement of the set X, we cannot say that one point is "less than" another, as there is no linear ordering for the domain of f_r. Therefore, we adapted a definition given by E.V. Huntington in [3] to produce the following definitions of cyclic and weak cyclic ordering.

Definition 2.2. *We say points $x_1, x_2, x_3 \in X$ are* cyclically ordered *if x_1 reaches x_2 before reaching x_3 by traveling clockwise, which we write as $[x_1\ x_2\ x_3]$.*

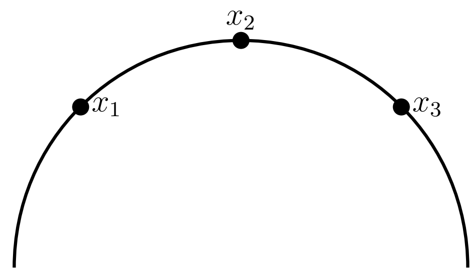

Figure 1 : Points x_1, x_2, and x_3 on a semicircle arranged so that $[x_1\ x_2\ x_3]$.

We can also have cyclically ordered intervals $A, B, C \subseteq S^1$ where by traveling clockwise, all points $x \in A$ reach all $x \in B$ before reaching any $x \in C$ where A, B and C are completely distinct intervals, which we write as $[A\ B\ C]$.

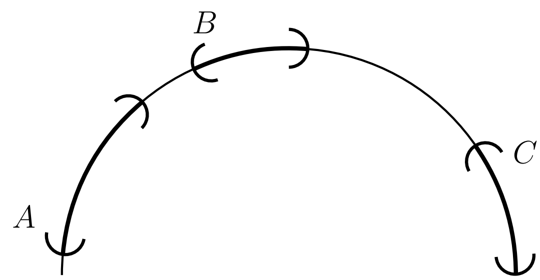

Figure 2 : Intervals A, B, and C on a semicircle arranged so that $[A\ B\ C]$.

The authors note in [1] that the function f_r preserves weak cyclic ordering on points in S^1. We will use this fact throughout this paper, so we give a definition for weak cyclic ordering here for completeness. First, we will define the relation represented by $\mathrel{\mathring{=}}$.

Definition 2.3. *We define the relation $x_1 \mathrel{\mathring{=}} x_2$ to mean that for some $x_1, x_2, x_3 \in X$ we have either $[x_1\ x_2\ x_3]$ or $x_1 = x_2$.*

Then, we say points $x_i \in X$ are weakly cyclically ordered if there exist some $x_m, x_n \in X$ such that $[x_0 \cdots x_m \mathrel{\mathring{=}} x_n \cdots x_i]$.

When talking about a finite dynamical system, the notion of periodicity arises naturally since all points eventually become periodic. We say a point $x \in X$ is *periodic* if there exists an $i > 0$ such that $f_r^i(x) = x$, and we call $\{f_r^i(x) | i \geq 0\}$ the *periodic orbit*. The *length* of a periodic orbit is the smallest $i \geq 1$ such that $f_r^i(x) = x$. In this paper, we also study the term "winding number" as defined in [1]. Before giving this definition, we present the following terms for ease of explanation.

Definition 2.4. *We say that a point x is* hit *in a sequence of iterates $\{y, f(y), \ldots, f^n(y)\}$ if there exists a point $f^i(y)$ such that $f^i(y) = x$ with $0 < i \leq n$.*

Definition 2.5. *We say x is* passed *if the arc along S^1 between any two points in a sequence of iterations of f_r hits or travels over x.*

Definition 2.6. *A point x is* skipped *if it is passed without being hit by the function f_r.*

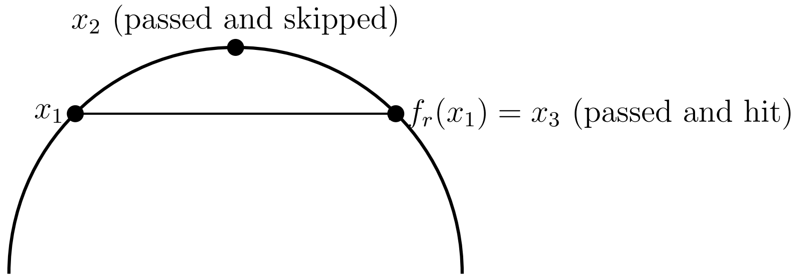

Figure 3 : Points passed, hit, and skipped on a semicircle with $[x_1\ x_2\ x_3]$ and $f_r(x_1) = x_3$.

From the definition of skipped we have Lemma 2.7, which we use to prove Lemma 2.9.

Lemma 2.7. *Let A be the periodic orbit generated by a point $x_0 \in X$. An equal number of points from A are skipped between any two consecutive iterates $f_r^k(x_0)$ and $f_r^{k+1}(x_0)$.*

Proof. Assume there exists an orbit A with points x_0 and $f_r(x_0)$. Cyclically order the orbit, and suppose that there are points $x_1, x_2, \ldots, x_{m-1} \in (x_0, f_r(x_0))_{S^1} \cap A$ where $x_1, x_2, \ldots, x_{m-1}$ represents all points in $(x_0, f_r(x_0))_{S^1} \cap A$.

Then there are exactly $m-1$ points $f_r^n(x_1), f_r^n(x_2), \ldots, f_r^n(x_{m-1}) \in (f_r^n(x_0), f_r^{n+1}(x_0)) \cap A$ for any $n \in \mathbb{N}$ since weak cyclic ordering is preserved as noted in [1]. Note that the ordering is strong because in a periodic orbit x_i maps to a unique $f_r(x_i)$. Therefore if there are $m-1$ points skipped between one pair of consecutive iterates, then there are $m-1$ points skipped between all pairs of consecutive iterates. □

We now return to define winding number as given in [1] using the terminology introduced in Definitions 2.4, 2.5, and 2.6.

Definition 2.8. *The* winding number $wn(x_0, \ldots, f_r^n(x_0))$ *of a sequence of points $x_0, \ldots, f_r^n(x_0) \in S^1$ is the number of times x_0 is passed when iterating through the sequence $x_0, f_r(x_0), \ldots, f_r^n(x_0)$ for $x_0 \in X$. Since S^1 has unit circumference, we have $wn(x_1, \ldots, x_s) = \sum_{i=1}^{s} \vec{d}(x_i, x_{i+1})$ where $x_{s+1} = x_1$.*

This definition implies a periodic orbit with length ℓ has winding number $w = \mathrm{wn}(x, f_r(x), \ldots, f_r^{\ell-1}(x))$. It is easy to check that given a periodic orbit with length ℓ and winding number w, it must be that $w < \ell$ and w and ℓ are relatively prime, as stated in [1]. The authors also stated that an orbit with cyclically ordered points implies that $f_r(x_i) = x_{(i+w) \bmod \ell}$ for all i. We expand upon this with the following Lemma.

Lemma 2.9. *If the vertices of a periodic orbit are cyclically ordered as $x_0, \ldots, x_{\ell-1}$, then we have that w is the winding number if and only if $f_r(x_i) = x_{(i+w) \bmod \ell}\ \forall\ i$ and $w < \ell$.*

Proof. We will first prove ($\Longrightarrow$). Assume a cyclically ordered periodic orbit $x_0, \ldots, x_{\ell-1}$ has length ℓ and winding number w. Then we have $w < \ell$. We also have that ℓw is the total number of passes for all points in ℓ iterations of the map of f_r, and ℓ is the number of hits in ℓ iterations. Thus we have $\ell w - \ell$ is the number of skips for all points in ℓ iterations. By Lemma 2.7 the number of skips between two consecutive iterations must be equal, so dividing the number of skips in ℓ iterations by the number of iterations gives the number of skips in a single iteration, $w - 1$.

No point can be skipped more than once in an iteration, because $r \leq 1$. Applying the function to x_i we then skip points $x_{i+1}, \ldots, x_{i+w-1}$ to land on $f_r(x_i) = x_{i+w \bmod \ell}$. Note we use mod ℓ otherwise we would not be contained within ℓ points.

We will now prove ($\Longleftarrow$). Assume $f_r(x_i) = x_{(i+m) \bmod \ell}$ and $m < \ell$ and for the sake of contradiction that $m \neq w$. Note that from ($\Longrightarrow$) we have that the winding number w is added to the index of refraction in each iteration. Then simultaneously m is added to the index in each iteration and w is added to the index in each iteration. Also note that $m \neq w + k\ell$ for any $k \in \mathbb{N}$ because $m < \ell$. Then for any i we have that $f_r(x_i) = x_{(i+m) \bmod \ell} \neq x_{(i+w) \bmod \ell}$ so it must be that $m = w$. □

It is clear that, depending on the arrangement of $X \subseteq S^1$ where $|X|$ is fixed and the value of r, more than one periodic orbit can form. This leads to the following definition.

Definition 2.10. *Two distinct orbits A and B are* interleaved *if for all $x, x' \in A$ there exists a $y \in B$ such that $[x\ y\ x']$.*

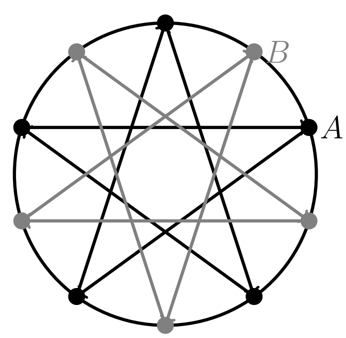

Figure 4 : Ten equidistant points on S^1 with $r = \frac{1}{2}$ and A and B interleaved.

The authors in [1] stated the following Lemma.

Lemma 2.11. *Any two periodic orbits are interleaved.*

It was noted in [1] that Lemma 2.11 implies that all periodic orbits of $f_r : X \to X$ have the same length ℓ and winding number w.

When comparing cyclic dynamical systems, we draw upon the definition of isomorphism.

Definition 2.12. *We say two systems* (f_r, X) *and* (f'_r, X') *are* isomorphic *if there exists a* $T : X' \to X$ *such that* $f_r \circ T = T \circ f'_r$.

The authors defined in [1] a regular cyclic dynamical system Reg_n^k for some $0 \le k < n$ where the domain is $X = \{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n-1}{n}\} \subseteq S^1$ and $r = \frac{k}{n} + \epsilon$ for some $0 < \epsilon < \frac{1}{n}$. They noted that all points of the regular system are periodic and given that $d = \gcd(k, n)$, Reg_n^k has d periodic orbits with length $\frac{n}{d}$ and winding number $\frac{k}{d}$. The core of a cyclic dynamical system is defined as the union of all periodic points in the system, which leads to Lemma 2.13, as stated in [1] .

Lemma 2.13. *The core of any cyclic dynamical system is isomorphic to* Reg_n^k *for some* $0 \le k < n$.

Given this knowledge, we can show results for cyclic dynamical systems generated by f_r in general while mainly focusing on equidistant arrangements.

It was noted in [1] that q-swift points imply relationships between r and the length and winding number of an orbit. We use an extension of q-swift points in our main theorem and include the original definition from [1] for reference.

Definition 2.14. *For* $r = \frac{p}{q}$, *a point* $x \in X$ *is* q-swift *if* $wn(x, f_r(x), \ldots, f_r^q(x)) = p$ *and the open arc* $(f_r^q(x), x)_{S^1}$ *does not contain any point of* X.

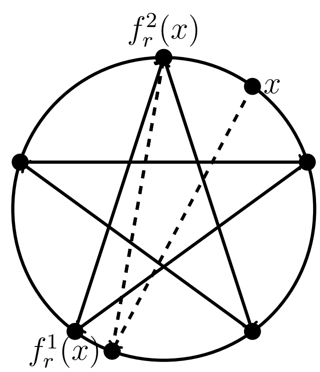

Figure 5 : Visualization of f_r where $r = \frac{p}{q}$ with $p = 1$ and $q = 2$ making x a q-swift point.

The following implication is taken from [1] and is included here for reference purposes.

Lemma 2.15. *Suppose* $r = \frac{p}{q}$ *and* $f_r : X \to X$ *contains a* q*-swift point* x. *Then the point* $f_r^q(x)$ *is periodic, and the system has a single periodic orbit whose winding number* w *and length* ℓ *satisfy* $\ell p - wq = 1$.

3. Implications of q_d-swift Points

After noting that a q-swift point implies one periodic orbit from Lemma 2.15 we consider what sort of point might imply d periodic orbits.

Definition 3.1. *Given $x_0 \in X$, define $X' \subseteq X$ to be the set of all iterates of x_0 and let X' be cyclically ordered. For $r = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and $f_r^q(x_0) = x_i$ we say x_0 is q_d-*swift *if the following conditions are met:*

1. *$wn(x_0, f_r(x_0), \ldots, f_r^q(x_0)) = p$*
2. *the open arc $(x_{(i+d-1) \bmod \ell}, x_0)_{S^1}$ contains exactly $d-1$ points, all of which are periodic and are not in X'.*

We provide an example of a q_d-swift point in Figure 6.

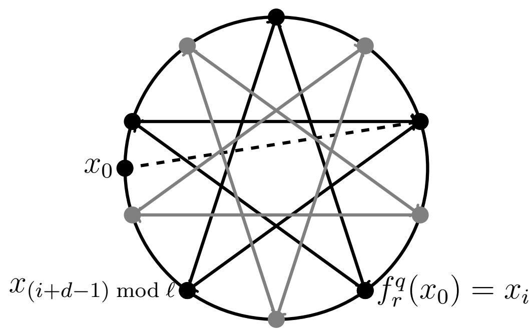

Figure 6 : Visualization of f_r where $r = \frac{p}{q}$ with $p = 4$ and $q = 9$ making x_0 a q_d-swift point with $d = 2$.

Lemma 3.2. *Suppose $r = \frac{p}{q}$ and $f_r : X \to X$ contains a q_d-swift point x. Then the first periodic point clockwise from or equal to x is a forward iterate of x.*

Proof. Let there be a q_d-swift point $x_0 \in X$. Define $X' \subseteq X$ to be the set of all iterates of x_0 and let X' be cyclically ordered as $X' = \{x_0, x_1, \ldots x_m\}$.

Case 1: If x_0 is periodic, then x_0 is a forward iterate of x_0 by definition.

Case 2: If x_0 is not periodic, then we want to show that the first periodic point clockwise from x_0 is a forward iterate of x_0. Let c_0 be the first periodic point clockwise from x_0 in X' and note $x_0 \neq c_0$. Define $X'' \subseteq X'$ to be the set of all iterates of c_0 and let X'' be cyclically ordered as $X'' = \{c_0, c_1, \ldots c_n\}$. Note that because x_0 maps into X'' there must be some k and c_α such that $f_r^k(x_0) = f_r^k(c_\alpha)$. Note also that $[x_0\ c_\alpha\ f(x_0)]$. If there is any point (periodic or non-periodic) y with $[x_0\ y\ c_\alpha]$ then by the preservation of cyclic ordering $f_r^k(x_0) = f_r^k(y) = f_r^k(c_\alpha)$ so y cannot be a periodic point. Then c_α is the first periodic point clockwise of x_0 and $c_\alpha \in X''$ so the first periodic point clockwise or equal to x_0 is a forward iterate of x_0.

Therefore the first periodic point clockwise from or equal to x is a forward iterate of x. $\square$

Lemma 3.3. *Let there be a q_d-swift point x_0 and define $x_{(i+d-1) \bmod \ell}$ as in Definition 3.1. Then $x_{(i+d-1) \bmod \ell} \neq x_0$.*

Proof. Assume for the sake of contradiction that $x_{(i+d-1) \bmod \ell} = x_0$. By the definition of q_d-swift, $x_{(i+d-1) \bmod \ell}$ is the first point counter-clockwise of x_0 that is in X'. Thus x_0 is a fixed point and cannot be q_d-swift, so it must be that $x_{(i+d-1) \bmod \ell} \neq x_0$ in general. □

Theorem 3.4. *Suppose $r = \frac{p}{q}$ and $f_r : X \to X$ contains a q_d-swift point x. Then $x_{(i+d-1) \bmod \ell}$ is periodic and the system has d periodic orbits whose winding number w and length ℓ satisfy $\ell p - wq = d$.*

Proof. Let there be a q_d-swift point $x_0 \in X$. Define $X' \subseteq X$ to be the set of all iterates of x_0, and let X' be cyclically ordered as $X' = \{x_0, x_1, \dots x_m\}$. Set $f_r^q(x_0) = x_i$. Let c_0 be the first periodic point equal to or clockwise from x_0. Note from Lemma 3.2 $c_0 \in X'$. Define $X'' \subseteq X'$ to be the set of all iterates of c_0 and let X'' be cyclically ordered as $X'' = \{c_0, c_1, \dots c_n\}$. Set $f_r^q(c_0) = c_j$. By definition of q_d-swiftness $X' \cap (x_{(i+d-1) \bmod \ell}, x_0)_{S^1} = \emptyset$, and from Lemma 3.3 we have $x_{(i+d-1) \bmod \ell} \neq x_0$ in general. Then $[x_0 \stackrel{\circ}{=} c_0 \stackrel{\circ}{=} x_{(i+d-1) \bmod \ell}]$. We want to show that $c_{(j+d-1) \bmod \ell}$ and c_j are consecutive points in X'' with exactly $d-1$ points between them.

Case 1: If $x_{(i+d-1) \bmod \ell} = c_{(j+d-1) \bmod \ell}$ it must be $c_{(j+d-1) \bmod \ell} \in [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$.

Case 2: If $x_{(i+d-1) \bmod \ell} \neq c_{(j+d-1) \bmod \ell}$ we want to show that it is still the case that $c_{(j+d-1) \bmod \ell}$
$\in [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$. Note that if $x_0 = c_0$, then $f_r^q(x_0) = f_r^q(c_0)$ and $X' = X''$, which implies $x_{(i+d-1) \bmod \ell} = c_{(j+d-1) \bmod \ell}$. Then $x_{(i+d-1) \bmod \ell} \neq c_{(j+d-1) \bmod \ell}$ implies $x_0 \neq c_0$ so x_0 is not periodic and $x_0 \neq c_{(j+d-1) \bmod \ell}$. Thus either $[c_0 \; c_{(j+d-1) \bmod \ell} \; x_{(i+d-1) \bmod \ell} \stackrel{\circ}{=}]^{(1)}$ or $[c_{(j+d-1) \bmod \ell} \; c_0 \stackrel{\circ}{=} x_{(i+d-1) \bmod \ell}]^{(2)}$.

Note if (1) then by the preservation of cyclic ordering $[c_0 \; x_0 \; c_{(j+d-1) \bmod \ell} \; x_{(i+d-1) \bmod \ell} \stackrel{\circ}{=}]$ but then $[x_0 \; c_{(j+d-1) \bmod \ell} \; c_0]$, and c_0 not is the first periodic point equal to or clockwise from x_0. Then if $x_{(i+d-1) \bmod \ell} \neq c_{(j+d-1) \bmod \ell}$, (2) must be true, so $c_{(j+d-1) \bmod \ell} \in [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$. Thus $[x_0 \stackrel{\circ}{=} c_0 \stackrel{\circ}{=} x_{(i+d-1) \bmod \ell}] \implies c_{(j+d-1) \bmod \ell} \in [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$ from Cases 1 and 2. Note by our choice of c_0 the only possible periodic points in $X' \cap [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$ are in $X' \cap [x_{(i+d-1) \bmod \ell}, x_0)_{S^1}$. By q_d-swiftness $x_{(i+d-1) \bmod \ell}$ is first point counter-clockwise of x_0 in X', and the only point of X' in the arc $[x_{(i+d-1) \bmod \ell}, x_0)_{S^1}$ is $x_{(i+d-1) \bmod \ell}$. We know $c_{(j+d-1) \bmod \ell}$ is a periodic point in $X' \cap [x_{(i+d-1) \bmod \ell}, c_0)_{S^1}$, so $c_{(j+d-1) \bmod \ell} = x_{(i+d-1) \bmod \ell}$ and $x_{(i+d-1) \bmod \ell}$ is periodic. Notice that $c_{(j+d-1) \bmod \ell}$ and c_0 are two consecutive periodic points in X'', and there are no periodic points in $(x_0, c_0)_{S^1}$. Thus, there are exactly $d-1$ periodic points interleaved between $c_{(j+d-1) \bmod \ell}$ and c_0. Therefore from Lemma 2.11 there must be d periodic orbits.

By Lemma 2.13, a system with d orbits of length ℓ and winding number w has a core that is isomorphic to $\mathrm{Reg}_{d\ell}^{dw}$. Setting $c_0 = 0$ implies $f_r^q(c_0) = \frac{\ell-d}{\ell}$. Also note that $f_r^q(c_0)$ is obtained from c_0 in q steps and $\mathrm{wn}(c_0, f_r(c_0), \dots, f_r^q(c_0)) = p$.Therefore $q\frac{w}{\ell} = p - \frac{d}{\ell}$ and $\ell p - wq = d$. □

Corollary 3.5. *Suppose $f_r : X \to X$ has orbits of length ℓ and contains an q_d-swift point x. Then for the number of orbits d it must be that $\ell > d$.*

Proof. Suppose $f_r : X \to X$ has orbits of length ℓ and contains an q_d-swift point x. Assume for the sake of contradiction that $\ell \leq d$ where d is the number of orbits. From Theorem 3.4 we have that $\ell p - wq = d$. Note that $\frac{d}{\ell} \geq 1$ so $q\frac{w}{\ell} = p - \frac{d}{\ell} \leq p - 1$. However, $q\frac{w}{\ell} \leq p - 1$ implies that in q iterations of the function from the point c the winding number is more than a full wind around the circle from p, so $\text{wn}(c, f_r(c), \ldots, f_r^q(c)) \leq p - 1$. Then $\text{wn}(c, f_r(c), \ldots, f_r^q(c)) = p$ and $\text{wn}(c, f_r(c), \ldots, f_r^q(c)) < p$, a contradiction. Therefore it must be that $\ell > d$. □

4. Expansions on Equidistant Arrangements

Adamaszek, Adams, and Motta proved in [1] that the core of any finite cyclic dynamical system (f_r, X) is isomorphic to some regular system Reg_n^k. Given the significance of equidistant arrangements, we study them more deeply here. Specifically, we focus on a single orbit on an equidistant arrangement. We do not adopt the Reg_n^k notation from [1] when we discuss equidistant arrangements, since we fix only the arrangement of the points of the domain and not the value of r (and thus the system itself).

Let $x_0 \in X$ be periodic and assume that $f_r(x_0)$ generates a unique periodic orbit on an equidistant arrangement. Recall from Section 3 that $X' \subseteq X$ is defined as the set of all iterates of x_0. Suppose that $X' = X$, i.e. the only points of the domain are those in the periodic orbit of x_0, and let X' be cyclically ordered. Then, the orbit has length ℓ and thus it must be that $\vec{d}(x_i, x_{i+1}) = \frac{1}{\ell}$. We ask the following: What if we perturb the arrangement by inserting a finite number of points? Then, how many distinct systems are possible? We answer these questions in the following section after providing some instrumental terminology and Lemmas.

From [1], we know that given a periodic orbit with winding number w and length ℓ, it must be that $w < \ell$ and w and ℓ are relatively prime. We show that any $w \in \mathbb{N}$ which satisfies these two restrictions is then a satisfactory winding number. That is, there exists an r which yields an orbit with that specific w.

Lemma 4.1. *For a unique orbit with length ℓ on an equidistant arrangement, if $y \in \mathbb{N}, y < \ell$, and y, ℓ are relatively prime, then there exists an r where $0 < r \leq 1$ such that the winding number of the orbit is y.*

Proof. Let $x_0 \in X$ be periodic and assume that $f_r(x_0)$ generates a unique periodic orbit on an equidistant arrangement. Suppose that $X' = X$, i.e. the only points of the domain are those in the periodic orbit of x_0, and let X' be cyclically ordered. Then, the orbit has length ℓ and thus it must be that $\vec{d}(x_i, x_{i+1}) = \frac{1}{\ell}$. Suppose $y \in \mathbb{N}, y < \ell$, and y, ℓ are relatively prime (since from [1] we know that y and ℓ must be relatively prime if y is to be the winding number).

Let $r = \frac{y+1}{\ell}$. Note that $\frac{y+1}{\ell} \leq 1$, since $y < \ell$ by assumption. We have that $f_r(x_0) \in X \cap [x_0, x_0 + r)_{S^1}$, where $x_0 + r$ is reached by traveling $\frac{y+1}{\ell}$ around the circle. Then, x_0 will be mapped to the counter-clockwise closest element to $x_0 + r$ (by definition of the function), reached by traveling $\frac{y}{\ell}$ around the circle since the elements are each $\frac{1}{\ell}$ apart. Thus, x_0 is mapped to x_y, since $\vec{d}(x_0, x_y) = \frac{y}{\ell}$.

By Lemma 2.7, the same number of points are skipped in each iteration in a periodic orbit. Then, $y - 1$ points will be skipped for each iteration in the orbit generated from $f_r(x_0)$. Therefore, by Lemma 2.9, y is the winding number. □

We also show which values of r are allowable for a single orbit on an equidistant arrangement with fixed length ℓ and winding number w.

Lemma 4.2. *For a single orbit on an equidistant arrangement with fixed length ℓ and winding number w, it must be that $\frac{w}{\ell} < r \leq \frac{w+1}{\ell}$.*

Proof. Let $x_0 \in X$ be periodic and assume that $f_r(x_0)$ generates a unique periodic orbit on an equidistant arrangement. Suppose that $X' = X$, i.e. the only points of the domain are those in the periodic orbit of x_0, and let X' be cyclically ordered. Then, the orbit has length ℓ and thus it must be that $\vec{d}(x_i, x_{i+1}) = \frac{1}{\ell}$. Fix a winding number w, where $w \in \mathbb{N}, w < \ell$, and w, ℓ are relatively prime.

Note that if $r \leq \frac{w}{\ell}$, then $\vec{d}(x_0, f_r(x_0)) < \frac{w}{\ell}$, and $w - 1$ points will not have been skipped in one iteration, i.e. w is not the winding number - a contradiction. We have that if $\frac{w+1}{\ell} < r$, then $\vec{d}(x_0, f_r(x_0)) > \frac{w+1}{\ell}$, and more than $w - 1$ points will have been skipped in one iteration, i.e. w is not the winding number - a contradiction. Therefore, we have $\frac{w}{\ell} < r \leq \frac{w+1}{\ell}$. □

Given our expansions on single orbits on equidistant arrangements, we move to explore perturbations of such arrangements. We show that, given a single orbit on an equidistant arrangement with fixed length and winding number, if we add a finite number of points between two consecutive points of the arrangement, $(x_j, x_{(j+1) \bmod \ell})_{S^1}$, a finite number of systems result. We then comment on the generalization of this result given the isomorphic relation between the core of certain cyclic dynamical systems and regular systems comprised of a single orbit on an equidistant arrangement.

First, we will define the term "itinerary", which we will make use of in our proof of Theorem 2.

Definition 4.3. *We say a list of forward and inverse iterates of f_r for a point $x_i \in X$ is the* itinerary *for x_i.*

We say two systems are “distinct” if they have distinct itineraries for at least one point. By this, we do not mean to say the point must be placed in the same spot on S^1 in both systems; it just needs to be in the same cyclically ordered spot in both systems. Notice that two isomorphic systems may have distinct itineraries and thus be distinct in this sense. We provide the following example for clarity.

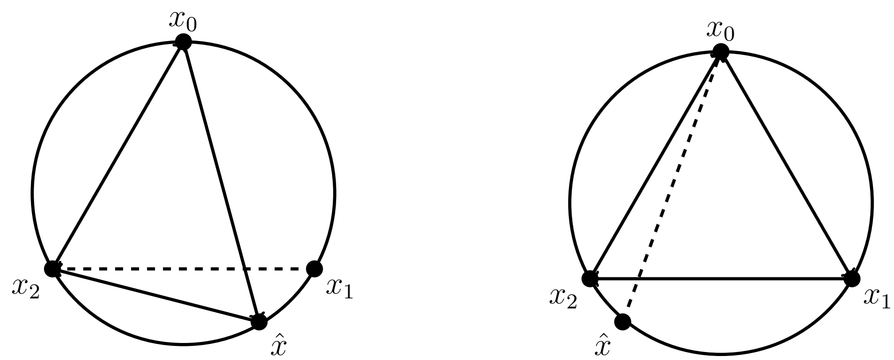

Figure 7 : Systems A (left) and B (right) with $r = \frac{1}{2}$. Note that systems A and B are isomorphic, but have distinct itineraries for each point and thus are distinct as we are defining the term.

Theorem 4.4. *For a single orbit on an equidistant arrangement with fixed length ℓ and winding number w, if we place a finite number of cyclically ordered points $\hat{x}_0, \dots, \hat{x}_{k-1}$ between two consecutive points of the arrangement $x_j, x_{(j+1) \mod \ell}$ where $j < \ell$, then $(k+1)^2$ distinct systems result.*

Proof. Let $x_0 \in X$ be periodic and assume that $f_r(x_0)$ generates a unique periodic orbit on an equidistant arrangement. Suppose that $X' = X$, i.e. the only points of the domain are those in the periodic orbit of x_0, and let X' be cyclically ordered. Then, the orbit has length ℓ and thus it must be that $\vec{d}(x_i, x_{i+1}) = \frac{1}{\ell}$. Fix a winding number w such that $w \in \mathbb{N}, w < \ell$, and w, ℓ are relatively prime. Place a finite number of points between two consecutive points of X such that we have $[x_j \; \hat{x}_0 \; \hat{x}_1 \cdots \hat{x}_{k-1} \; x_{(j+1) \mod \ell}]$.

By Lemma 4.2, we have $\frac{w}{\ell} < r \leq \frac{w+1}{\ell}$. Thus, only $x_{(j-w) \mod \ell}$ is mapped into $[x_j, x_{(j+1) \mod \ell})_{S^1}$, since all $x_i \in X$ are equidistant. So, $x_{(j-w) \mod \ell}$ is mapped to either x_j or one of the inserted points $\hat{x}_0, \dots, \hat{x}_{k-1}$ and is the only element which could do so. Therefore, there are $k+1$ possible itineraries for $x_{(j-w) \mod \ell}$. Note that for all other $x_i \in X'$, we have $f_r(x_i) = x_{(i+w) \mod \ell}$.

By Lemma 4.2 and because we fixed an equidistant arrangement, there are only two elements which the k inserted points could map to, $x_{(j+w) \mod \ell}$ and $x_{(j+1+w) \mod \ell}$. So, we have that some number α of the k points are mapped to $x_{(j+w) \mod \ell}$ and $k - \alpha$ are mapped to $x_{(j+1+w) \mod \ell}$.

Note if $[x_j \; \hat{x}_n \; \hat{x}_s]$, then we cannot have that $\hat{x}_n \to x_{(j+1+w) \mod \ell}$ and $\hat{x}_s \to x_{(j+w) \mod \ell}$, since the points are cyclically ordered. Thus, the itineraries of the k points must respect the cyclic ordering, i.e. we have that the α points which are mapped to $x_{(j+w) \bmod \ell}$ can be written $\hat{x}_0, \ldots, \hat{x}_{\alpha-1}$ and the $k-\alpha$ points mapped to $x_{(j+1+w) \bmod \ell}$ can be written $\hat{x}_\alpha, \ldots, \hat{x}_{k-\alpha}$. Then, we have there are $k+1$ possible values of α: $0, \ldots, k$. Therefore, combined with the $k+1$ possible itineraries for $x_{(j-w) \bmod \ell}$, we have $(k+1)(k+1) = (k+1)^2$ distinct systems. It is left as an exercise for the reader to check that each system could form. □

We now seek to comment on our ability to generalize these results, which leads us to the following corollary:

Corollary 4.5. *We have that the $(k+1)^2$ distinct systems are isomorphic to one of two systems with equidistant arrangements.*

Proof. Let $x_0 \in X$ be periodic and assume that $f_r(x_0)$ generates a unique periodic orbit on an equidistant arrangement. Suppose that $X' = X$, i.e. the only points of the domain are those in the periodic orbit of x_0, and let X' be cyclically ordered. Then, the orbit has length ℓ and thus it must be that $\vec{d}(x_i, x_{i+1}) = \frac{1}{\ell}$. Fix a winding number w such that $w \in \mathbb{N}, w < \ell$, and w, ℓ are relatively prime. Place a finite number of points between two consecutive points of X such that we have $[x_j \ \hat{x}_0 \ \hat{x}_1 \cdots \hat{x}_{k-1} \ x_{(j+1) \bmod \ell}]$.

We have that, if $x_{(j-w) \bmod \ell}$ maps to one of the k inserted points, $\hat{x}_s$, and $\hat{x}_s$ maps to the closer of the two elements it could be sent to, $x_{(j+w) \bmod \ell}$, then the resulting system is isomorphic to the original system, since we have essentially replaced x_j with $\hat{x}_s$. Also note that if $f_r(x_{(j-w) \bmod \ell}) = x_j$, then the resulting system is trivially isomorphic to the original one.

However, if $x_{(j-w) \bmod \ell}$ maps to one of the k inserted points, $\hat{x}_s$, and $f_r(\hat{x}_s) = x_{(j+1+w) \bmod \ell}$, then the resulting system is isomorphic to a new system with an equidistant arrangement with at most $\ell - 1$ terms in the orbit, since we skip $x_{(j+1) \bmod \ell}$ and replace x_j with $\hat{x}_s$.

□

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p-adic Geometry in Two Dimensions

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Abstract: Let p be prime. We generalize the notion of p-adic absolute value to define a new distance on $\mathbb{Q}^2$. We compare the geometric behaviors between traditional Euclidean geometry and our extended p-adic geometry. As an application, we look at triangles in $\mathbb{Q}^2$ and study their properties.

1. Introduction

Throughout, let p be a prime. For any integer $n \neq 0$, we define its p-adic valuation, denoted $v_p(n)$, to be the largest power of p which divides n. We define $v_p(0) = \infty$ because $p^k \mid 0$ for all k. Let's denote the p-adic valuation of a number n by $v_p(n)$.

Example 1.1. *We have that*

$$v_3(63) = v_3(3^2 \cdot 7) = 2$$
$$v_7(63) = v_7(3^2 \cdot 7) = 1$$
$$v_5(63) = v_5(3^2 \cdot 7) = 0.$$

We can extend p-adic valuation to the rationals with the following definition.

Definition 1.2. *For $\frac{a}{b} \in \mathbb{Q}$, define $v_p(\frac{a}{b}) = v_p(a) - v_p(b)$.*

Note that $v_p(\frac{a}{b})$ is well-defined.
Thus, p-adic valuation is a function

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}.$$

Example 1.3. *We have that*

$$v_2\left(\frac{7}{8}\right) = v_2(7) - v_2(8) = 0 - 3 = -3.$$

Lemma 1.4. *For $a, b \in \mathbb{Q}$, $v_p(a + b) \geq min\{v_p(a), v_p(b)\}$ with equality if and only if $v_p(a) \neq v_p(b)$.*

For $x \in \mathbb{Q}$, we define its p-adic absolute value, denoted $|x|_p$, to be the following:

$$|x|_p = \frac{1}{p^{v_p(x)}}, x \in \mathbb{Q},$$

and given $x, y \in \mathbb{Q}$, we define their p-adic distance by

$$|x - y|_p = \frac{1}{p^{v_p(x-y)}}.$$

Example 1.5. *We have that*

$$|17 - 9|_2 = |8|_2 = \frac{1}{2^{v_2(8)}} = \frac{1}{2^3} = \frac{1}{8}.$$

Lemma 1.6. *[1] For all $a, b, c \in \mathbb{Q}$, we have that*

1. $v_p(ab) = v_p(a) + v_p(b)$
2. $|ab|_p = |a|_p|b|_p$
3. $|a|_p \geq 0$ *with* $|a|_p = 0$ *if and only if* $a = 0$
4. $v_p(a - b) \geq min\{v_p(a), v_p(b)\}$ *with equality when* $v_p(a) \neq v_p(b)$
5. $|a - b|_p \leq |a - c|_p + |b - c|_p$ *(The Triangle Inequality).*

Lemma 1.7. *For all $a, b \in \mathbb{Q}$, we have $|a - b|_p = |b - a|_p$.*

Proof. Let $a, b \in \mathbb{Q}$. Using Lemma 1.6 (2), we have

$$\begin{aligned} |a - b|_p &= |(-1)(b - a)|_p \\ &= |(-1)|_p|b - a|_p \\ &= |b - a|_p. \end{aligned}$$

□

The focus of this work is to introduce the notion of p-adic distance in the two-dimensional plane, and to study its properties.
Given the point $(x, y) \in \mathbb{Q}^2$ we define its p-adic absolute value to be

$$|(x, y)|_p = \sqrt{(|x|_p)^2 + (|y|_p)^2}.$$

Additionally, we can define the distance between two points, $(x_1, y_1), (x_2, y_2) \in \mathbb{Q}^2$ as follows:

$$d_p((x_1, y_1), (x_2, y_2)) = |(x_1, y_1) - (x_2, y_2)|_p = \sqrt{(|x_2 - x_1|_p)^2 + (|y_2 - y_1|_p)^2}.$$

Example 1.8. *We have*

$$\begin{aligned} d_3((3, 8), (6, 9)) &= |(3, 8) - (6, 9)|_3 \\ &= \sqrt{(|6 - 3|_3)^2 + (|9 - 8|_3)^2} \\ &= \sqrt{(|3|_3)^2 + (|1|_3)^2} \\ &= \sqrt{\left(\frac{1}{3^1}\right)^2 + \left(\frac{1}{3^0}\right)^2} \\ &= \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}. \end{aligned}$$

It is natural to consider whether statements 2, 3, and 5 of Lemma 1.6 hold in two dimensions. We start with:

Lemma 1.9. *There exist $u, v \in \mathbb{Q}^2$ such that $|uv|_p \neq |u|_p|v|_p$, and Lemma 1.6 (2) does not hold true in $\mathbb{Q}^2$.*

Proof. We provide an example to illustrate that equality does not always hold. Let $u = (0, 1), v = (1, 0)$. Thus

$$|uv|_p = |(0, 0)|_p = 0,$$

while

$$|u|_p|v|_p = |(1, 0)|_p|(0, 1)|_p = 1 \cdot 1 = 1.$$

□

The following shows that Lemma 1.6 (3) holds true in $\mathbb{Q}^2$.

Lemma 1.10. *For $u \in \mathbb{Q}^2, |u|_p \geq 0$, with $|u|_p = 0$ if and only if $u = (0, 0)$.*

Proof. Let $u = (x, y)$. We have $|u|_p = \sqrt{\left(\frac{1}{p^{v_p(x)}}\right)^2 + \left(\frac{1}{p^{v_p(y)}}\right)^2} \geq 0$. Because neither of the summands within our square root can be negative, if $|u|_p = 0$, then $\frac{1}{p^{v_p(x)}} = \frac{1}{p^{v_p(y)}} = 0$. This happens if and only if $v_p(x) = v_p(y) = \infty$, or $x = y = 0$. □

To show Lemma 1.6 (5), we require

Lemma 1.11. *Let $u, v \in \mathbb{Q}^2$. We have that $d_p(ku, kv) = |k|_p d_p(u, v)$.*

Proof. Let $u = (x_1, y_1), v = (x_2, y_2) \in \mathbb{Q}^2$. Using Lemma 1.6 we have

$$\begin{aligned}
d_p(ku, kv) &= \sqrt{\left(\frac{1}{p^{v_p(kx_1 - kx_2)}}\right)^2 + \left(\frac{1}{p^{v_p(ky_1 - ky_2)}}\right)^2} \\
&= \sqrt{\left(\frac{1}{p^{v_p(k(x_1 - x_2))}}\right)^2 + \left(\frac{1}{p^{v_p(k(y_1 - y_2))}}\right)^2} \\
&= \sqrt{\left(\frac{1}{p^{v_p(k) + v_p(x_1 - x_2)}}\right)^2 + \left(\frac{1}{p^{v_p(k) + v_p(y_1 - y_2)}}\right)^2} \\
&= \sqrt{\left(\frac{1}{p^{v_p(k)}}\right)^2 \left(\left(\frac{1}{p^{v_p(x_1 - x_2)}}\right)^2 + \left(\frac{1}{p^{v_p(y_1 - y_2)}}\right)^2\right)} \\
&= \frac{1}{p^{v_p(k)}} \sqrt{\left(\frac{1}{p^{v_p(x_1 - x_2)}}\right)^2 + \left(\frac{1}{p^{v_p(y_1 - y_2)}}\right)^2}.
\end{aligned}$$

Therefore,

$$d_p(ku, kv) = |k|_p \sqrt{\left(\frac{1}{p^{v_p(x_1 - x_2)}}\right)^2 + \left(\frac{1}{p^{v_p(y_1 - y_2)}}\right)^2} = |k|_p d_p(u, v).$$

□

2. Lines

As we observe points in $\mathbb{Q}^2$, we can consider the notion of lines, which are defined as the connection between any two points a, b.

Definition 2.1. *Given points $u, v, w \in \mathbb{Q}^2$, the line through u, w is the set*

$$L_{\{u,w\}} = \{v : (d_p(u, v) + d_p(u, w) - d_p(v, w))(d_p(u, v) + d_p(v, w) - d_p(u, w))(d_p(u, w) + d_p(v, w) - d_p(u, v)) = 0\}.$$

This is intuitively the same as how lines are defined in Euclidean Geometry. Lines exist between any three points u, v, w such that the sum of the distance from u to v and the distance from v to w is equal to the distance from u to w. Our definition of a line encompasses this concept without loss of generality.
The main result of this section is

Theorem 2.2. *Given distinct* $u, v, w \in \mathbb{Q}^2$, *we have that* $d_p(u, w) < d_p(u, v) + d_p(v, w)$. *That is, all lines contain exactly two points.*

To prove this we need the following result in $\mathbb{Q}$.

Lemma 2.3. *For all* $a, b, c \in \mathbb{Q}$, *we have that* $|a - b|_p^2 \leq |b - c|_p^2 + |a - c|_p^2 + 2\lambda$, *where* $\lambda = (|b - c|_p)(|a - c|_p)$ *is nonnegative.*

Proof. We have from Lemma 1.6 (5) that $|a - b|_p \leq |b - c|_p + |a - c|_p$. Squaring both sides gives

$$(|a - b|_p)^2 \leq (|b - c|_p)^2 + (|a - c|_p)^2 + 2(|b - c|_p|a - c|_p)$$

and $\lambda := 2(|b - c|_p|a - c|_p)$ is nonnegative. Therefore

$$(|a - b|_p)^2 \leq (|b - c|_p)^2 + (|a - c|_p)^2 + 2\lambda.$$

□

Lemma 2.4. *Given* $u, v, w \in \mathbb{Q}^2$, *we have that* $d_p(u, w) \leq d_p(u, v) + d_p(v, w)$.

Proof. Let $u = (x_1, y_1), v = (x_2, y_2), w = (x_3, y_3) \in \mathbb{Q}^2$. We want to show that

$$d_p(u, w) \leq d_p(u, v) + d_p(v, w).$$

We can rewrite the above inequality as

$$\sqrt{(|x_1 - x_3|_p)^2 + (|y_1 - y_3|_p)^2} \leq \sqrt{(|x_1 - x_2|_p)^2 + (|y_1 - y_2|_p)^2} + \sqrt{(|x_2 - x_3|_p)^2 + (|y_2 - y_3|_p)^2}.$$

Throughout, let $x_{ij} = |x_i - x_j|_p$, $y_{ij} = |y_i - y_j|_p$. Thus we have

$$\sqrt{(x_{13})^2 + (y_{13})^2} \leq \sqrt{(x_{12})^2 + (y_{12})^2} + \sqrt{(x_{23})^2 + (y_{23})^2}.$$

Since all of the components are nonnegative, we can square both sides of this inequality and obtain

$$(x_{13})^2 + (y_{13})^2 \leq (x_{12})^2 + (y_{12})^2 + (x_{23})^2 + (y_{23})^2 + 2\sqrt{r}$$

where $r = ((x_{12})^2 + (y_{12})^2)((x_{23})^2 + (y_{23})^2)$.
From Lemma 2.3, we have that

$$(x_{13})^2 \leq (x_{12})^2 + (x_{23})^2 + 2\lambda_x$$
$$(y_{13})^2 \leq (y_{12})^2 + (y_{23})^2 + 2\lambda_y$$

where $\lambda_x = (x_{12})(x_{23}),\ \lambda_y = (y_{12})(y_{23})$.
Thus

$$(x_{13})^2 + (y_{13})^2 \leq (x_{12})^2 + (y_{12})^2 + (x_{23})^2 + (y_{23})^2 + 2(\lambda_x + \lambda_y)$$

We claim that $\lambda_x + \lambda_y \leq \sqrt{r}$, from which the lemma will follow.
So we wish so show that

$$(x_{12})(x_{23}) + (y_{12})(y_{23}) \leq \sqrt{\left((x_{12})^2 + (y_{12})^2\right)\left((x_{23})^2 + (y_{23})^2\right)}.$$

Squaring both sides gives

$$\begin{aligned}
(x_{12})^2(x_{23})^2 + (y_{12})^2(y_{23})^2 + 2(x_{12})(x_{23})(y_{12})(y_{23}) &\leq ((x_{12})^2 + (y_{12})^2)((x_{23})^2 + (y_{23})^2) \\
(x_{12})^2(x_{23})^2 + (y_{12})^2(y_{23})^2 + 2((x_{12})(x_{23})(y_{12})(y_{23})) &\leq (x_{12})^2(x_{23})^2 + (y_{12})^2(y_{23})^2 \\
&\quad + (y_{12})^2(x_{23})^2 + (x_{12})^2(y_{23})^2 \\
2(x_{12})(x_{23})(y_{12})(y_{23}) &\leq (y_{12})^2(x_{23})^2 + (x_{12})^2(y_{23})^2 \\
\frac{2}{p^{v_p(x_1-x_2)+v_p(x_2-x_3)+v_p(y_1-y_2)+v_p(y_2-y_3)}} &\leq \frac{1}{p^{2(v_p(y_1-y_2)+v_p(x_2-x_3))}} \\
&\quad + \frac{1}{p^{2(v_p(x_1-x_2)+v_p(y_2-y_3))}}.
\end{aligned}$$

If we let $m = v_p(y_1 - y_2) + v_p(x_2 - x_3), n = v_p(x_1 - x_2) + v_p(y_2 - y_3)$, then

$$\frac{2}{p^{m+n}} \leq \frac{1}{p^{2m}} + \frac{1}{p^{2n}}$$

$$2 \leq \frac{p^n}{p^m} + \frac{p^m}{p^n}$$

which is necessarily true, because $\frac{p^m}{p^n}$ and $\frac{p^n}{p^m}$ are multiplicative inverses, and we have that $x + \frac{1}{x} \geq 2$ for all $x > 0$.
Therefore we have that the claim holds and the theorem is true.

□

Proof of Theorem 2.2. Let $u, v, w \in \mathbb{Q}^2$. Assume without loss of generality that $d_p(u, w) \geq d_p(u, v)$ and $d_p(u, w) \geq d_p(w, v)$. It suffices to show that $v \notin L_{\{u,w\}}$. By Lemma 1.11 we can replace u, v, w with $u-u, v-u, w-u$. By a change of variables, we may assume that $u = (0, 0)$ and $w = (1, 0)$.
In order for $v \in L_{\{u,w\}}$, we must have $d_p(u, w) = d_p(u, v) + d_p(v, w)$, that is,

$$1 = d_p(u, v) + d_p(v, w).$$

Let $v = (0, p^t b)$ for $t, b \in \mathbb{Z}$, where $v_p(b) = 0$. We have

$$\sqrt{\frac{1}{p^{2t}}} + \sqrt{1 + \frac{1}{p^{2t}}} > 1,$$

thus $v \notin L_{u,w}$.
Let $v = (p^s a, 0)$ for $s, a \in \mathbb{Z}$, where $v_p(a) = 0$. Then

$$\sqrt{\frac{1}{p^{2s}}} + \sqrt{\frac{1}{p^{2v_p(p^s a - 1)}}} > 1,$$

thus $v \notin L_{u,w}$.
Let $v = (p^s a, p^t b)$ for $s, t, a, b \in \mathbb{Z}$, where $v_p(a) = v_p(b) = 0$. We will investigate different cases of s and t.
Suppose $s, t > 0$. Then

$$\sqrt{\frac{1}{p^{2s}} + \frac{1}{p^{2t}}} + \sqrt{1 + \frac{1}{p^{2t}}} > 1.$$

thus $v \notin L_{u,w}$.

Suppose $s \leq 0$ or $t \leq 0$. Then

$$\sqrt{\frac{1}{p^{2s}} + \frac{1}{p^{2t}}} + \sqrt{\frac{1}{p^{2v_p(p^s a - 1)}} + \frac{1}{p^{2t}}} \geq 1 + \sqrt{\frac{1}{p^{2t}}} > 1,$$

thus $v \notin L_{u,w}$.
Thus, there are no possible values of s, t such that a point $v = (p^s a, p^t b) \in L_{\{u,w\}}$ exists. Therefore $L_{\{u,w\}} = \{u, w\}$. □

3. Triangles

We will now consider triangles in $\mathbb{Q}^2$ under p-adic geometry.

Definition 3.1. *Given three points $a, b, c \in \mathbb{Q}^2$, an equilateral triangle is defined by the set $\{a, b, c\}$ when $d_p(a, b) = d_p(a, c) = d_p(b, c)$.*

Lemma 3.2. *Given $(x_1, x_2) \in \mathbb{Q}^2$, if $|(x_1, x_2)|_p = 1$ then either $x_1 = 0$ or $x_2 = 0$, and the other component has valuation 0.*

Proof. Let $(x_1, x_2) \in \mathbb{Q}^2$ and $|(x_1, x_2)|_p = 1$. Then,

$$1 = |(x_1, x_2)|_p = \sqrt{p^{-2v_p(x_1)} + p^{-2v_p(x_2)}}.$$

Squaring both sides gives us

$$1 = p^{-2v_p(x_1)} + p^{-2v_p(x_2)},$$

from which it follows that one of the summands is 1 and the other is 0. Therefore one of x_1, x_2 has valuation ∞, corresponding to $x_1 = 0$ or $x_2 = 0$, and the other has valuation 0. □

Theorem 3.3. *Given $u = (x_1, y_1), v = (x_2, y_2), w = (x_3, y_3) \in \mathbb{Q}^2, \{u, v, w\}$ define an equilateral triangle if and only if $|x_1 - x_2|_p = |x_2 - x_3|_p = |x_3 - x_1|_p$ and $|y_1 - y_2|_p = |y_2 - y_3|_p = |y_3 - y_1|_p$.*

Proof. If $|x_1 - x_2|_p = |x_2 - x_3|_p = |x_3 - x_1|_p$ and $|y_1 - y_2|_p = |y_2 - y_3|_p = |y_3 - y_1|_p$, then it is easy to show that the resulting triangle is equilateral. We now show that any equilateral triangle satisfies these identities.
We can assume by Lemma 1.11 that $u = (0, 0)$. It suffices to consider equilateral triangles of side length 1. By Lemma 3.2, we can assume that $v = (x_2, 0), w = (0, y_3)$ for $x, y \in \mathbb{Q}$. (Note that if $v = (x_2, 0), w = (x_3, 0)$, then $|x_1 - x_2|_p = |x_2 - x_3|_p = |x_3 - x_1|_p$ and $|y_1 - y_2|_p = |y_2 - y_3|_p = |y_3 - y_1|_p$ and we are done.)
We have that $d_p(u, v) = d_p(u, w) = 1$. But

$$d_p(v, w) = \sqrt{p^{-2v_p(x_2)} + p^{-2v_p(y_3)}} = \sqrt{1 + 1} = \sqrt{2} \neq 1.$$

Thus there does not exist an equilateral triangle of this form. □

4. References

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Abstracts

Terrence Brown
Department of Mathematics
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Title: The Five Color Theorem

My Senior Seminar project is on a topic in Graph Theory, known as the Five Color Theorem. In it, we will learn some introductory definitions used all the time in Graph Theory like a graph, vertex, edge, and degree. From there, we will learn about different types of graphs and we will discuss Euler's Formula. Then we will discuss a concept called coloring, briefly discuss the Four Color Theorem and finally, we will state and prove the main theorem, the Five Color Theorem.

Cedric Campbell and Jamar Posey
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Birmingham-Southern College
Title: Colley Matrix with NEW lead change factor

We will examine the Colley Matrix and use its system to rank college football. This matrix system has played a pivotal role for years in the ranking of this college sport. In conjunction with the Colley Matrix, we have added another dimension to strengthen its rankings method. To fortify the matrix, we will use lead changes to help ensure a team has the proper ranking from week to week. We believe lead changes play a big role in determining if a team should receive a better ranking than another, based on a quality win or horrible loss. Our results will be compared to the previous football season rankings as well as the new College Football Playoff Committee. Our ranking system will be a new and improved measure that takes out bias in relation to college football standings. It will be an actual representation of how a team performs week in and week out.

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Department of Mathematics
Georgia Institute of Technology
Title: Word Trees and Bubble Trees: A Comparison of Data Visualization Methods For Search Optimization of Research Paper Databases

Modern research institutions have large amounts of data to process, organize, and search. Searching through data individually is time-consuming and inefficient, so it is necessary to create tools to improve this process. Current methods of database searching raise issues such as yielding results with jargon words or little relation to the query. These methods also frequently lack a useful priority ranking for each result. Furthermore, modern research archives such as PubMed limit their search to title-only, abstract-only, keyword-only, or figure caption text. This widely hinders the depth of the search, leaving multiple potential papers out of the results. Even if full-text search were an option, this would still leave the persisting problem of false-positive (incorrect indication that a word is present) and false-negative (incorrect indication that a word is absent) results. This study approaches these issues with data visualization techniques, since we sought to improve not only the results, but also the way the user interacts with them.

Through this study on data visualization (DV) of biomedical research text, we developed two methods of solving these issues by presenting search result data in an intuitive, functional format. The first uses the Word Tree (WT), which displays multiple branching relationships between the most frequently used words/phrases. The second is the Bubble Tree (BT), which provides a keyword's related words within a file to specify users' search. Both methods provide an additional data point, the "weight" of the word or phrase, which describes how often the specified text is used in the file.

We investigate and compare these two methods of DV: WT and BT, to determine which is better at reducing the number of jargon results and conveying the relevance of each result. Compared to existing methods, our proposed DV technique provides more accurate full-text database search results in addition to the ability to further narrow down results with a graphical display. We will demonstrate two DVs with "astrocyte" (a biomedical term) as the keyword.

Firstly, we produced customized word lists using statistical analysis and machine learning processes based on 200 papers about "astrocytes" as our training set. Secondly, we collected and summarized words from existing biomedical ontologies, such as RGDWeb, MeSH, and cellML. Each DV technique illustrates how creating a DV out of research paper text generates a dynamic map of a keyword to its related words/phrases, letting the user interact with the results beyond merely clicking on a top result. These techniques also mitigate the issue of false-positive results, since false-positives will have lower weights - saving the user time and effort.

Both methods assist researchers' database searching, and optimize search results. Our data visualization strategy demonstrates a new approach to the issues that searching currently presents, which would give greater flexibility, accuracy, priority, and function to database search results.

Keyanna Davis
Department of Mathematics and Computer Science
Albany State University
Title: Applications of Parabolic Partial Differential Equations to Food Processing, Food Preservation and Cylinder Manufacturing

Parabolic partial differential equations belong to one of the major classes on partial differential equations, with the other two classes being elliptic partial differential equations and hyperbolic partial differential equations. In this paper, we study analytical solutions of a class of parabolic partial differential equations with Initial-boundary conditions. Furthermore, we apply the methods studied to application problems: food processing, food preservation, and cylinder manufacturing.

Keyanna Davis, Jessica Haynes, Katlynn Stodghill and Thomas Floyd
Department of Mathematics and Computer Science
Albany State University
Title: Analytical Case Studies of Incarceration in Dougherty County, GA

Incarceration, in the United States, is the common form of punishment for felonies and lower level offenses. With only representing 4.4 percent of the world population, the United States houses 22 percent of the world's offenders. This research project will analyze the statistics of the incarcerations in the county jail in Dougherty County, GA from 2010-15. From the analysis of the criminal statistics, the correlation between ages, genders, and race will be determined.

Tanner Dixon
Department of Mathematics
Birmingham-Southern College
Title: Analyzing the Flow Free Problem and the Decomposition of Graphs into Paths

In this paper we generalize the puzzle game application Flow Free into a mathematical $n \times n$ graph. Applying certain chosen in-game restrictions and rules, we analyze when weak and strong solutions exist. We decompose the original graph into useful paths in order to properly analyze particular and general graph constructions.

William Dula
Department of Mathematics
Morehouse College
Title: NSFD Discretization for the Lotka-Volterra Predator-Prey Mathematical Model

The Lotka-Volterra model was the first attempt at deriving a mathematical representation of the interaction between predator and prey populations. While it is known not to provide an accurate model of this class of population dynamics, it does give several important insights into predator-prey phenomena. Since the two coupled non-linear differential equations can not be solved explicitly in terms of the elementary functions, numerical solutions can be obtained by means of a variety of standard numerical integration techniques. Our major goal is to construct a discretization based on the Non-Standard Finite Difference Methodology of Ronald E. Mickens. We investigate this scheme and compare its mathematical properties to those of corresponding original differential equations. Particular features of interest include the number, location, and linear stability of the fixed-points; and the positivity of the solutions. Our calculations show that the NSFD discretization is dynamically consistent with the important features of the Lotka-Volterra system.

Brennan Farmer
Department of Mathematics and Computer Science
Albany State University
Title: Buying Versus Renting a House

In this project, we visit various mortgage banks serving in Southwest Georgia to find out the lowest mortgage interest rate offered to a loan seeker with a given credit score. We collect the available combinations of the interest rates and service charges for a loan. We initially go with the 15-yr fixed mortgage plan to win the lowest mortgage interest rate. Later, we assume that monthly payment is higher than the minimum mortgage as per the 15-year fixed mortgage plan and prepare a dynamical system, consisting of difference equations, corresponding to the interest rate. We solve and analyze the dynamical system, display the amount of loan at various months under a higher fixed payment scheme. We display the results in the spread sheet as well as in the graphs. At the end, we compare the benefits of buying versus renting a house by taking into consideration of the real world situations, and provide a strong argument supported by our data.

Thomas G. Floyd
Department of Mathematics and Computer Science
Albany State University
Title: Using Agricultural and Fixed-Income Investments to Enhance Economic Growth Of Early County, Georgia

For many years, counties in rural Georgia, including Early County, have been decreasing in population. This decrease has had immense negative economic impact on such counties: the tax base has decreased, other economic activities have dwindled. There is therefore the need to use statistical and financial mathematical tools to determine those attributes which could increase economic viability of such counties. Early County is on the Southwest quadrant of Georgia. It is mostly rural, with agriculture as the major source of economic income. The population of the county is about $11,713$, with one county high school. While many citizens embark on small agricultural productions for food and sustenance, Early County has some large agricultural companies and ranches, and there is room for young people to consider self-employment through agriculture and other fixed income investments. In this study, we present a statistical study on agriculture, time series graphs, and forecast based on available data. The role of agriculture and fixed income investments in the improvement of Early County's viability and growth is delineated.

Audrey Goodnight
Department of Mathematics
Agnes Scott College
Title: Cyclic Dynamical Systems

Our research concerns finite dynamical systems generated by a function f_r Defined by Adamaszek, Adams and Motta (2016). The function takes a set of points X on a circle of unit circumference to itself by mapping each x in X to the clockwise furthest point within a given arc length r with $0 < r \leq 1$. Although the eventual periodicity of any point in a finite dynamical system is well known, the specific behavior of this system in and before a periodic orbit presents an interesting avenue for further study. In particular, Adamaszek, Adams and Motta identified a special class of points denoted q-swift points. The existence of at least one q-swift point was found to imply a single periodic orbit in the entire system. We extended this definition to dq-swift points which were found to imply d periodic orbits.

Jesse Handlon and Austin Martin
Department of Mathematics
Birmingham-Southern College
Title: An Introduction to the Black-Scholes Formula

We introduce financial mathematics and examine the Black-Scholes formula. To understand the Black-Scholes formula one must understand its components. We then go through the process of pricing an option using the Black-Scholes formula, and give reasons to why an investor would consider purchasing options. We focus on developing mathematical intuition as a tool for stimulating further undergraduate interest in financial mathematics.

Tiffany Heppard, Kierra Goodwin, Brandi Sumter and Alston Rice
Department of Mathematics and Computer Science
Albany State University
Title: Numbers Don't Lie

The purpose of the research is to determine the future outcome of the NFL Draft. The project will take the data from 2000-2016 NFL Draft and apply several parameters which will produce the percentage of what schools produce the most draftees. Upon seeing which schools produce the most draftees, the researchers will apply another set of parameters to specifically find the trend of why schools have more draftees than others.

Dru Horne
Department of Mathematics
University of Georgia
Title: Counting Graph Derangements of the Ladder Graph L_n

Permutation and derangements on n letters are well understood, and these can be viewed as graph permutations and graph derangements on K_n. What can we say about a general simple graph G? How many graph permutations are there for G? How many graph derangement? In this talk, we will look at the simple graph L_n, the ladder graph, and count the number of graph derangements. In the process, we will generate a surprising sequence of numbers that is related to a well understood combinatorial sequence.

Qixuan Hou
Department of Industrial and Systems Engineering
Georgia Institute of Technology
Title: Modeling Distinct Flight Boarding Procedures

In order to achieve high aircraft utilization, commercial airlines has made efforts to improve its turnaround performance, which is measured by the time between an airplane's arrival and its departure. Passenger boarding is one of the many factors which determine turnaround time. We simulate distinct airline boarding procedures with Python, such as outside-in, random, rear to front, reverse pyramid, rotating zone, zone/block style, and also provide a mathematical model to measure the goodness of each procedure. By analyzing the model, we want to evaluate distinct boarding processes and to identify the most efficient boarding strategy. Ideally, with field observations and data analysis, a new procedure will be proposed to optimize the boarding time and improve turnaround performance.

Vernard Hurd and Alston Rice
Department of Mathematics and Computer Science
Albany State University
Title: Modeling Distinct Flight Boarding Procedures

In a previous study, we applied simple linear regression to the estimation of regression parameters in Agricultural data modeling. Monotonic nonlinear transformations such as logarithmic, inverse or exponential functions of independent and/or dependent variables may model predictions accurately for Fibonacci type growth patterns, but if the relationship between X and Y is not monotonic, a polynomial regression becomes more accurate. A polynomial has two or more terms. The polynomials we most often use in simple polynomial regression are the quadratic, and the cubic. With a quadratic, the slope for predicting Y from X changes direction once, with a cubic it changes direction twice.

In this study, we transition the Predictive model from Simple Linear Regression to Polynomial Regression in order to better estimate the curvilinear relationship between Land Crop Rent Value and Crop Yield Values.

Some of the key questions we ask and attempt to answer in this research are:

a) What does the scatter diagram of the land rental rate versus crop yield value data look like?

b) Is there evidence that the polynomial model (specifically the quadratic and cubic model) could be more accurate than the simple linear regression model in predicting regression parameters in Land Crop Rent Rates versus Crop Yield Values?

c) Is there evidence of significant curvature to the relationship between Land Crop Rent Rates and Crop Yield Values?

d) How do Linear and Polynomial Regression Estimates compare asymptotically?

e) Could a mix of the two methods produce even more accurate results?

Foster Johnstone
Department of Mathematics
Birmingham-Southern College
Title: Modeling Invasive Lionfish Growth and Diffusion

Over the years, invasive species have proven to cause harmful problems to Biological ecosystems. Coral reefs in the Western Atlantic and Caribbean have been plagued by one particularly harmful species, lionfish. Lionfish, native to the Indo-Pacific, were introduced off the coast of Florida in 1985. Since then, their population has increased at an alarming rate, and they have spread up the coastline of the Eastern United States, to islands in the Caribbean and throughout the Gulf of Mexico. In a report with CNN, Graham Maddocks, president and founder of the Ocean Support Foundation, states the lionfish invasion is probably the worst environmental disaster the Atlantic will ever face. The National Oceanic and Atmospheric Administration says that research has shown that a single lionfish can reduce the amount of native fish on a reef by up to 79 percent. Because of the abundance of prey, favorable climate and lack of natural predators, aside from humans, lionfish populations have exploded throughout the Caribbean and Western Atlantic. Understanding population dynamics for lionfish is important in addressing the threat they pose to these reefs. This paper discusses and models the growth and diffusion of lionfish throughout the Caribbean and Western Atlantic, making use of subpopulations in each area. There are seven subpopulations and each one is broken up into two parts, juveniles and adults. To control movement of lionfish between subpopulations and the adult and juvenile populations, a set of fourteen differential equations are used. Using this model can provide a better understanding of travel patterns and lead to solutions for controlling their spread.

Cynthia Kagambirwa
Department of Mathematics
Birmingham-Southern College
Title: One-sided incomplete information in a three-person bargaining game

Most games are comprised of repetition, this is the foundation of our research. We analyze a three-person bargaining game with one-sided incomplete information. This research studies a game in which two informed players repeatedly alternate making offers to divide a pot of size T among the three players. One player is uncertain of the payoff function or the size T. We use Rubinstein's model on two-person alternating-offer bargaining game with two sided incomplete information to generalize the three-person situation; we will investigate the short-term and long-term effects of one-sided incomplete information in the game. Meanwhile, we consider how players with extra information use the additional information to their advantage and how the uninformed player should gather information based on past plays or conflicts. Generally, repetition will drive players to cooperative agreements despite conflicts of interests among players.

Jasmine Key
Department of Mathematics
Birmingham-Southern College
Title: Invariant Elements in Permutation Groups to Generate Magic

Magic tricks are grounded in distorting the spectator's reality. The magician is able to provide an unexpected result by misguiding the focus of a trick and leaving important elements invariant throughout the trick. Through the exploration of invariant elements, we will embark on a mathematical analysis of a card trick in the permutation group S_5. This card trick allows the spectator the freedom to alter the positions of the cards but consistently produces the predicted result using invariant elements.

Leandre Kibeho
Department of Mathematics
Morehouse College
Title: Matroids

Matroids were introduced by Whitney in 1935 to try to capture abstractly the essence of dependence. In this paper, I will present the background history of matroids and their definition. I will discuss the different types of matroids and their properties and also the proof of those properties. I will also discuss the basis, rank and circuit of matroids and their properties. In this paper, I will also present the application of matroids in different fields of optimization mostly in construction of the matroidal network coding capacity of networks under various network coding schemes. In this paper, I am going to address the problem with the approach of matroidal networks. In this approach, I will prove the converse of the theorem which states that, if a network is scalar-linearly solvable, then it is a matroidal network associated with a representable matroid over a finite field.

Catherine Konde, Maurice Howard, Ne'kera Smith and Vreonna Strong-Berlin
Department of Mathematics
Albany State University
Title: A Comparative Analysis of Retention and Graduation in the University System of Georgia

This research project seeks to investigate the increase or decrease of students from the University System of Georgia (USG). The data that will be used for this research focuses on the first-time, full-time freshmen from the fall 2009 cohort. The research will collect, compile, and examine the data on the graduation rates of the undergraduates from the

fall 2009 cohort, who completed their degree program within four years, five years, and six years. Similarly, data showing the retention rates of first-time, full-time undergraduates, who continued their education at the same school within a six-year period (2009-2015), will be compiled and examined. After comparing the data, we will discuss how the institutions have fared in retaining and graduating their students.

Peter Mi
Department of Mathematics
Birmingham-Southern College
Title: The Classification of Timbres via Overtone Sequences

Scientists commonly use “three elements of sound” to describe a certain sound: amplitude, frequency and timbre. Timbre, also known as tonal color, has the most complicated mechanism of the three, and it depends on the overtone sequence. Using software that can produce and edit sounds, I classify the timbres in a mathematical way, looking for mathematical description of musical categories.

Emily Piff
Department of Mathematics
Agnes Scott College
Title: P-adic Geometry in Two Dimensions

Let p be prime. We generalize the notion of p-adic valuation to define a new distance on $Q \times Q$. We compare the geometric behaviors between traditional Euclidean geometry and our extended p-adic geometry. As an application, we look at triangles in $Q \times Q$ and study their properties.

Mohlomi Taoana
Department of Mathematics
Morehouse College
Title: Devil Facial Tumor Disease (DFTD)

Devil Facial Tumor Disease (DFTD) is one of the three rarest transmissible cancers affecting the last surviving carnivorous marsupial known as Tasmanian Devils, which are native to the island of Tasmania. We develop a mathematical model of the spread of the disease throughout the population of Tasmania and develop methods to curb the spread of the disease. In doing so, we model the cancer similarly to an infectious disease by exploring the commonly used Susceptible Infected and Recovered (SIR) model. Since recovery in cancer is not possible, the R term is disregarded thereby changing the model into an SI model. We then explore methods to decrease the spread of DFTD. We separate potion of the infected population annually for two reasons, namely; to reduce the frequency of contact between infected and susceptible, secondly, we allow for breeding to happen among the quarantine infected devils. A mathematical simulation of this strategy portrays a decline in prevalence of DFTD with disease extinction expected within ten to fifteen years. These results suggest a path to disease eradication; hence the model can be used to in conjunction with the current strategies to curb the prevalence of DFTD.

Tracey Vu
Department of Mathematics
Birmingham-Southern College
Title: Mathematical Strategies for the Game 2048

The game 2048 is a new game that was first introduced as an online game by Gabriele Cirulli on March 9, 2014. The game is now a popular app for mobile devices. 2048 involves powers of 2 and swiping motions to move and combine the numbers on the grid with a goal in mind to create a space on the grid containing 2048. In this paper, matrices and counting functions will be observed in order to create a strategy in order to win the game. It is proven in this paper that a smaller 2x2 version of the game and its moves can be fully represented by matrices and transition matrices. It is also proven that a player can at least reach a space containing 8 in every game.

Mary-Stewart Wachter
Department of Mathematics
Birmingham-Southern College
Title: Simulating the Rotation of a Tornado

In the field of Meteorology, there is still much to be discovered about tornadoes. There is a considerable gap in research for how they are formed. To contribute to this, we must fully understand the fundamentals of how the atmosphere is governed by particular equations. Narrowing our search, we will analyze the Navier-Stokes equations for the velocity components of wind speed. Using applied numerical analysis, we will enter these equations into Matlab. By fixing certain components of the Navier-Stokes equations, we will see if pressure alone is able to overcome the dispersion factor and cause rotation, thus simulating a simplified model of a tornadoes rotational aspects.

Brishanti Weaver
Department of Mathematics and Computer Science
Albany State University
Title: Solving Systems of Equations using Mathematical Software

Consider the following system of equations:

$$x + y - 3z = -8 \quad \text{and} \quad 2x + 5y - 7z = -22.$$

We can easily solve this system through hand-held computations: $x = 2$, $y = -1$ and $z = 3$. The task becomes more challenging when (i) the degree goes higher and (ii) the system goes bigger. Now consider the following system of equations used to find a special rational function called the Belyi function:

$$\begin{aligned}
c(a + 2h + 4) &= 0\\
-be^2 + cd^2 &= 0\\
-c(4a - 2g - h^2 + 6) &= 0\\
2c(3a + f + gh + 2) &= 0\\
2cdf + 2be + e^2 &= 0\\
2cdg + cf^2 - b - 2e &= 0\\
-c(4a - 2d - 2fh - g^2 + 1) &= 0\\
2cdh + 2cfg + ac + 1 &= 0
\end{aligned}$$

This is a system of 8 equations and 8 unknowns. It is too difficult to solve this system manually. This is one example to reinforce the necessity of computer software to solve mathematical problems. Our Maple implementation returns the following solution to the above system:

$$a = -8/7,\ b = 25/21,\ c = 343/192,\ d = -1000/2401$$
$$e = 25/49,\ f = 340/343,\ g = -15/49,\ h = -10/7.$$

In this presentation, we will implement efficient techniques to solve systems of equations like above arising from real world applications using mathematical software like Maple.